QUADRATIC DIFFERENTIALS AS STABILITY CONDITIONS: COLLAPSING SUBSURFACES

ANNA BARBIERI, MARTIN MÖLLER, YU QIU, AND JEONGHOON SO

Dedicated to Bernhard Keller on the occasion of his sixtieth birthday

Abstract. We introduce a new class of triangulated categories, which are Verdier quotients of 3-Calabi-Yau categories from (decorated) marked surfaces, and show that its spaces of stability conditions can be identified with moduli spaces of framed quadratic differentials on Riemann surfaces with arbitrary order zeros and arbitrary higher order poles.

A main tool in our proof is a comparison of two exchange graphs, obtained by tilting hearts in the quotient categories and by flipping mixed-angulations associated with the quadratic differentials.

Contents

1. Introduction

The notion of stability conditions on a triangulated category $\mathcal{D}$ was introduced by Bridgeland in [Bri07]. Since then, the stability space $\operatorname{Stab}\mathcal{D}$, which as a set consists of Bridgeland stability conditions on $\mathcal{D}$, has played a major role in algebraic geometry, representation theory, mirror symmetry and some branches of mathematical physics, providing interesting synergies. By its very definition $\operatorname{Stab}\mathcal{D}$ comes with a $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-action, just as moduli spaces of framed abelian and quadratic differentials do.

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While the global structure of $\operatorname{Stab}\mathcal{D}$ as a complex manifold is still unknown in many cases, there are examples that are quite well understood. This includes for instance the case of the stability space of a class of three-Calabi-Yau (CY_3) categories constructed from the Ginzburg algebra of quivers with potentials, that are well known categories in representation and cluster theory. Inspired by the work of Gaiotto-Moore-Neitzke [GMN13], Bridgeland and Smith have shown in [BS15] that some moduli spaces of meromorphic quadratic differentials with simple zeros can be identified with those spaces of stability conditions, appropriately quotiented by the action of autoequivalences.

Our goal here is to generalize the Bridgeland-Smith correspondence to quadratic differentials with arbitrary higher order zeros. It implies studying another class of categories, which are related to the previous ones as quotients, but seem less well-behaved. Our motivation for this is two-fold.

Categorification. Spaces of quadratic differentials with higher order zeros arise when zeros collide. As such they form a subspace of the total space of quadratic differentials with no zero order condition, in fact a subspace locally cut out by linear conditions in period coordinates. In the spaces of abelian and quadratic differentials, the $\mathbb{R}$-linear submanifolds have received a lot of attention (see e.g. [Fil20] for a recent survey on the classification problem, however with focus on holomorphic differentials). Since these submanifolds admit an action of the universal cover of $\mathrm{GL}_2^+(\mathbb{R})$, a natural question is whether they all can be interpreted as spaces of stability conditions on an appropriate triangulated category.

More generally one can analyze the collision of zeros and poles, or even the collapse of a higher genus subsurface. Our main result gives an answer to the question how to interpret such collapses categorically. In a nutshell, collapses correspond to taking Verdier quotients.

Compactification. Spaces of stability conditions are typically non-compact, even after projectivization, and several strategies of compactification have recently been explored. Some of them are Thurston-type compactifications with real codimension one boundary ([BDL20], [KKO22]), some of them are partial compactifications ([Bol20], [BPPW22]). On the other hand, spaces of projectivized quadratic differentials have a compactification as smooth complex orbifolds (combine [BCGGM2] and [BCGGM1]) and in forthcoming work we will recast this compactification in terms of 'multi-scale stability conditions' for quiver CY_3 categories. Spaces of quadratic differentials with higher order zeros appear naturally as boundary strata in this compactification.

1.1. **The main result.** The combinatorics of a meromorphic quadratic differential q on a Riemann surface S is encoded in a weighted decorated marked surface $\mathbf{S_w}$, the real blow-up of S at the poles of q, see Section 3.2 for the definition. The pole orders are encoded in the markings of (a finite number of points in) the boundary components of $\mathbf{S_w}$, and usually hidden from notation. The weights $\mathbf{w}$ encode the tuple of orders of zeros of the differential. The horizontal trajectories of a generic q induce a tiling of $\mathbf{S_w}$ into polygons, whose number of edges depends on the orders of the zeros they contain.

The case of $\mathbf{S}_\Delta := \mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$ is the one originally considered by [BS15] and [KQ20]. It corresponds to quadratic differentials with simple zeros. These differentials induce a triangulation of $\mathbf{S}_\Delta$ to which, in turn, one associates a quiver with potential

(Q, W) and its Ginzburg (differential graded) algebra $\Gamma(Q, W)$. The triangulated category $\mathcal{D}_3(\mathbf{S}_\Delta)$ is defined as the perfectly valued derived category $\operatorname{pvd}(\Gamma(Q, W))$, that is the subcategory of $\mathcal{D}(\Gamma)$ of Γ-modules with finite dimensional total homology. The correspondence of [BS15; KQ20] can be restated as an isomorphism of complex manifolds

$$K : \mathrm{FQuad}^\circ(\mathbf{S}_\Delta) \to \mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}_\Delta)),$$

involving the moduli space of (Teichmüller-)framed quadratic differentials on $\mathbf{S}_\Delta$ and a connected component $\mathrm{Stab}^\circ(\mathcal{D})$ of the stability manifold of $\mathcal{D}_3(\mathbf{S}_\Delta)$.

Section 2 contains background material on Bridgeland stability conditions and quotient categories, as well as how to associate Ginzburg categories to quivers with potential. We summarize the notions of marked surfaces, weighted decorations and quadratic differentials in Section 3. The geometry of moduli spaces with all kinds of framings is recalled in Section 4. The previous results by [BS15; KQ20] together with mapping class group actions are restated in Theorem 7.1.

Consider now quadratic differentials with signature $\mathbf{w}$ different from the 'trivial' case $\mathbf{w} \equiv \mathbf{1}$ and their associated $\mathbf{S}_\mathbf{w}$. A weighted decorated marked surface with non-trivial weight can be obtained by collapsing a subsurface Σ in $\mathbf{S}_\Delta$, as we explain in Section 5. In such a case we denote it by $\overline{\mathbf{S}}_\mathbf{w}$. *In the whole paper the surface $\overline{\mathbf{S}}_\mathbf{w}$ has at least one boundary component (i.e. the quadratic differentials are meromorphic), there are no punctures (i.e. none of the marked points is a regular point of the quadratic differentials) and we disallow double poles and simple poles* to avoid several technicalities like working with cohomology valued in local systems (the space $\mathrm{Quad}^\heartsuit$ of [BS15]) and self-folded triangles in triangulations.

The main result of this paper is Theorem 7.2, stated in short form as follows:

Theorem 1.1. *There is an isomorphism of complex manifolds*

$$K : \mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_\mathbf{w}) \to \mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$$

between the principal part of the space of Teichmüller-framed quadratic differentials inducing the weighted decorated marked surface $\overline{\mathbf{S}}_\mathbf{w}$ and the principal part of the space of stability conditions on the Verdier quotient

$$\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}) := \mathcal{D}_3(\mathbf{S}_\Delta)/\mathcal{D}_3(\Sigma).$$

In this theorem the bullet points ('principal part') refer to a union of connected components, defined in Sections 5.4 and 6.3 respectively, and motivated below. Our results can most likely be extended to include punctures and small order poles with appropriate care. The case of holomorphic differentials is a whole different story, for which the recent categorification by Haiden ([Hai21]) could be the point of departure.

A given decorated marked surface $\overline{\mathbf{S}}_\mathbf{w}$ may be realized as the collapse of several different surfaces $\mathbf{S}_\Delta$ with simple weights: the case $g(\mathbf{S}_\Delta) = g(\overline{\mathbf{S}}_\mathbf{w})$ is always possible, $g(\mathbf{S}_\Delta) > g(\overline{\mathbf{S}}_\mathbf{w})$ is possible if the entries of $\mathbf{w}$ are large enough. Since the spaces of framed quadratic differentials do not depend on the collapse, Theorem 1.1 gives the realization of the same manifold M as $M \cong \mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ for different triangulated categories $\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})$. However the autoequivalences of $\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})$ detect those different realizations, just as the mapping class groups do on the topological side, see Section 6.4. On the side of framed differentials, every component of $\mathrm{FQuad}(\overline{\mathbf{S}}_\mathbf{w})$ is realized as a component of $\mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ for appropriate choices of initial triangulations. For spaces of stability conditions however we make no claims on the (non)-existence of spurious components of $\mathrm{Stab}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ not covered as target of our correspondence, just as this question is left undecided in [BS15] for simple zeros.

1.2. **Techniques.** The proof of Theorem 1.1, given in Section 7, shares with the original proof by Bridgeland and Smith the idea of extending a chamber-wise identification. The main differences are an explicit isomorphism of exchange graphs on both sides and a generalization of the method to extend beyond the 'tame locus', as we now explain.

Both $\mathrm{FQuad}(\overline{\mathbf{S}}_\mathbf{w})$ and $\mathrm{Stab}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ come with a natural chamber structure. In $\mathrm{FQuad}(\overline{\mathbf{S}}_\mathbf{w})$ the open chambers are given by quadratic differentials without horizontal saddle connections. The trajectory structure of the differential gives rise to an arc system that we call $\mathbf{w}$-*mixed-angulation*, generalizing the triangulations in the simple zero case. The preimage of the mixed-angulation under the collapse is called a *partial triangulation* of $\mathbf{S}_\Delta$. Adjacency of chambers is encoded by a notion of forward flip of the partial triangulation and leads to the definition of an exchange graph $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$. On the other side, $\mathrm{Stab}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ is also tiled in chambers identified by the heart of a bounded t-structure the stability conditions are supported on. The first step consists of studying and comparing these chamber structures.

Comparison of exchange graphs. We start from a distinguished heart of $\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})$ and need to consider the exchange graph $\mathrm{EG}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ whose vertices are hearts of bounded t-structures and whose arrows are simple tilts (recalled in Section 2). The idea is to relate (parts of) $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$ and $\mathrm{EG}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$. When $\mathbf{w} \equiv \mathbf{1}$ this is the relation between triangulations and finite hearts of bounded t-structures of a CY_3 Ginzburg category.

Recall that $\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})$ is by definition a Verdier quotient of a CY_3 category $\mathcal{D}_3(\mathbf{S}_\Delta)$. While a partial triangulation can always be refined to a triangulation, a general expectation is that *not* all hearts in $\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})$ arise as quotients of hearts in $\mathcal{D}_3(\mathbf{S}_\Delta)$. When they do, we call them hearts *of quotient type*. We restrict to the principal part $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ of $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$ whose vertices are those partial triangulations that can be refined to a successive flip of a triangulation $\mathbb{T}$ fixed once and for all. Correspondingly, $\mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ includes precisely hearts that are quotients of tilts of the heart of $\mathcal{D}_3(\mathbf{S}_\Delta)$ associated to $\mathbb{T}$ under the original ($\mathbf{w} \equiv \mathbf{1}$) correspondence. The definition and study of these graphs covers Sections 5 and 6 and leads to the isomorphism

$$\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w}) \cong \mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})), \tag{1.1}$$

stated as Theorem 6.9. It allows us to define the map K of Theorem 1.1 on the complement B_2 of the locus of differentials with more than one horizontal saddle connection.

The viewpoint of refining partial triangulations to triangulations makes it also clear why quotient categories naturally arise in this context: Any two triangulation refinements of a partial triangulation differ by successive flips in the additional edges and we show in Proposition 6.5 that the resulting quotient heart is independent of these choices. A consequence of the main result and (1.1) is that the union of $\mathbb{C}$-orbits of hearts of quotient type of $\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})$ form connected components of $\mathrm{Stab}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$.

Walls have ends. Finally we need extend $K|_{B_2}$ to all of $\mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$, which is stratified by the number of closed saddle connections and recurrent trajectories.

We generalize in Section 4.2 the argument in [BS15] that each component of all the higher order strata B_p of $\mathrm{FQuad}(\overline{\mathbf{S}}_{\mathbf{w}})$ has 'ends' where it locally does not disconnect the complement. Our argument gives an alternative proof that does not depend on case distinctions of local configurations of hat-homologous saddle connections. Those configurations probably become hard to list as the orders of zeros in $\mathbf{w}$ grow. As a downside, our approach avoids classifying the moduli spaces of objects in $\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})$ that are stable and of phase zero in a given stability condition σ, compare [BS15, Theorems 1.4 and 11.6]. Due to the connection with computing BPS-invariants in the CY_3 context, it seems interesting to analyze this further.

1.3. **The category $\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})$.** The category $\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})$ is defined as the quotient of a CY_3 triangulated category $\mathcal{D}(\mathbf{S}_\Delta) := \mathrm{pvd}(\Gamma(Q,W))$ by a subcategory of the same form $\mathcal{D}_3(\Sigma) := \mathrm{pvd}(\Gamma(Q_I, W_I))$, where (Q_I, W_I) is a subquiver of the quiver with potential (Q,W) defined by the combinatorial data of a quadratic differential. As opposed to $\mathcal{D}(\mathbf{S}_\Delta)$, the quotient category is in general not Calabi-Yau and not Hom-finite, yet we need to consider its bounded t-structures. Proposition 6.8 and Theorem 6.9 in Section 6, beyond proving the isomorphism of the graphs (1.1), tell us about the possibility to lift a simple tilt in the quotient $\mathrm{pvd}(\Gamma(Q,W))/\,\mathrm{pvd}(\Gamma(Q_I,W_I))$ to a simple tilt on $\mathrm{pvd}(\Gamma(Q,W))$ and viceversa. A comprehensive description of these categories and their t-structures will appear in a subsequent paper.

1.4. **Exchange graphs and connected components, examples.** The classification of connected components of spaces of abelian or quadratic differentials has attracted a lot of attention, and similarly the question whether spaces of stability conditions are connected is an important question in the topic. We give a short overview over the literature. For differentials, there are two classification questions. For (plain, unframed) differentials, the first result is by Kontsevich-Zorich ([KZ03]) for holomorphic abelian differentials, followed by Lanneau ([Lan08]) for holomorphic quadratic differentials. Boissy first classified components for meromorphic abelian differentials ([Boi15]). See work of Chen-Gendron [CG22] for the latest results. Equally interesting and challenging is the classification of (Teichmüller-)framed differentials, see [KQ20] for simple zero and higher order pole case and work of Walker ([Wal09]) and Calderon-Salter ([CS21]) for the latest results in the holomorphic case. In almost all known cases components are classified by spin invariants, hyperellipticity and torsion conditions in genus one, some low genus strata of quadratic differentials providing exceptions (see [Lan08] and also [CM14]).

For spaces of stability conditions the stability manifold is known to be connected (and simply connected) for instance for the bounded derived category of curves (Okada [Oka06] for genus $g = 0$ and Macrì [Mac07] for higher genus) or of some abelian surfaces and very general $K3$ surfaces ([HMS08]). An example for a non-connected space of stability conditions is given by Meinhardt and Partsch [MP14]. They study the quotient category $\mathcal{D}^b_{(1)}(X)$ of the bounded derived category $\mathcal{D}^b(X)$ on a smooth projective variety X with $\dim(X) \geq 2$ by the full subcategory of complexes of sheaves supported in codimension $c > 1$. The classification of components is based on computing $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ orbits of $\mathrm{Stab}(\mathcal{D}^b_{(1)}(X))$.

The walls-have-ends result Corollary 4.3 implies that connectivity of spaces of quadratic differentials is equivalent to the connectivity of the corresponding exchange graphs. Via our main theorem this gives a criterion to show if the spaces

$\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ are disconnected. In fact, Example 5.15 together with the isomorphism 1.1 and Theorem 1.1 shows disconnectivity for example already for the space of quadratic differentials with a zero of order three and a triple pole.

Corollary 1.2. *For a surface $\overline{\mathbf{S}}_{\mathbf{w}}$ of genus one with one boundary component and one zero with weight $\mathbf{w} = (3)$ the principal part $\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ is disconnected.*

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2. Preliminaries on categories and the stability manifold

In this section we set some notation and collect background material about stability conditions on triangulated categories, quotient categories, quivers with potential and the CY_3-categories associated to them. References are [HRS96; GM03; BBD82; Bri07; Bri09; Nee14; DWZ08; Kel11a].

Notation. We fix $\mathbf{k}$ an algebraically closed field for simplicity. All categories considered in this paper are $\mathbf{k}$-linear and all subcategories are full. For an additive category $\mathcal{C}$ with a subcategory (or set of objects) $\mathcal{B}$, we define

$$\mathcal{B}^{\perp_\mathcal{C}} := \{C \in \mathcal{C} : \mathrm{Hom}_\mathcal{C}(B, C) = 0 \;\forall\, B \in \mathcal{B}\},$$

and similarly ${}^{\perp_\mathcal{C}}\mathcal{B}$. We will omit the subscript $\mathcal{C}$ when there is no confusion. If $\mathcal{C}$ is triangulated, we denote by $\mathrm{thick}(\mathcal{B})$ the smallest thick additive full subcategory in $\mathcal{C}$ containing $\mathcal{B}$. It is triangulated. Moreover, for full subcategories $\mathcal{H}_1, \mathcal{H}_2$ of an abelian or a triangulated category $\mathcal{C}$, we let

$$\begin{aligned} \mathcal{H}_1 * \mathcal{H}_2 &:= \{M \in \mathcal{C} \mid \exists \text{ s.e.s or triangle } T \to M \to F \text{ s.t. } T \in \mathcal{H}_1, F \in \mathcal{H}_2\}, \\ \langle \mathcal{B} \rangle &:= \{M \in \mathcal{C} \mid \exists \text{ s.e.s or triangle } T \to M \to F \text{ s.t. } T, F \in \mathrm{Add}\,\mathcal{B}\}. \end{aligned}$$

Consequently $\mathcal{H}_1 * \mathcal{H}_2 \subset \langle \mathcal{H}_1, \mathcal{H}_2 \rangle \supset \mathcal{H}_2 * \mathcal{H}_1$. If $\mathcal{H}_1, \mathcal{H}_2$ satisfy $\mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2) = 0$, then we write $\mathcal{H}_1 \perp \mathcal{H}_2$ for $\mathcal{H}_1 * \mathcal{H}_2$.

A finite length abelian category will be said to be *finite* if it has finitely many simple objects. Throughout the paper we will use the following complex half-planes

$$\begin{aligned} \mathbb{H} &:= \left\{\rho e^{\pi i \theta} \,|\, \rho \in \mathbb{R}_{>0},\ 0 < \theta < 1\right\}, \\ \overline{\mathbb{H}} &:= \left\{\rho e^{\pi i \theta} \,|\, \rho \in \mathbb{R}_{>0},\ 0 < \theta \le 1\right\}. \end{aligned} \tag{2.1}$$

2.1. Structures on triangulated and abelian categories.

Here we collect some background material about bounded t-structures, stability conditions on triangulated categories, and quotients of abelian and triangulated categories.

Bounded t-structures and simple tilts. A *t-structure* on a triangulated category $\mathcal{D}$ is the torsion part of a torsion pair (so that $\mathcal{D} = \mathcal{P} \perp \mathcal{P}^\perp$) satisfying $\mathcal{P}[1] \subset \mathcal{P}$. The t-structure is said to be *bounded* if $\mathcal{D} = \cup_{m \in \mathbb{Z}} \mathcal{P}[m] \cap \mathcal{P}^\perp[-m]$. The *heart* of a bounded t-structure $\mathcal{P} \subset \mathcal{D}$ is the full subcategory $\mathcal{H} = \mathcal{P} \cap \mathcal{P}^\perp[1]$, which is abelian. Denote by $K(\mathcal{H}) \simeq K(\mathcal{D})$ their Grothendieck groups. The bounded t-structure and its heart determine each other uniquely and hence we will use them interchangeably.

Given a torsion pair $(\mathcal{T},\mathcal{F})$ in the abelian heart of a bounded t-structure $\mathcal{H}=\mathcal{T}\perp\mathcal{F}$, there is a new heart $\mu^\sharp_{\mathcal{F}}\mathcal{H}:=\mathcal{T}\perp_{\mathcal{D}}\mathcal{F}[1]$, known as *forward tilt with respect to* $(\mathcal{T},\mathcal{F})$ of $\mathcal{H}$, see e.g. [HRS96]. Obviously, tilting commutes with autoequivalences, i.e., for any $\Phi\in\operatorname{Aut}(\mathcal{D})$,

$$\Phi\left(\mu^\sharp_{\mathcal{T}}(\mathcal{H})\right) \;=\; \mu^\sharp_{\Phi(\mathcal{T})}\Phi(\mathcal{H})\,. \tag{2.2}$$

A forward tilting $\mathcal{H}\to\mathcal{H}^\sharp$ is simple if the corresponding torsion free class $\mathcal{F}$ is generated by a simple S of $\mathcal{H}$, i.e. $\mathcal{F}=\langle S\rangle$. For a finite heart $\mathcal{H}$ with a rigid simple S the simple forward tilt with respect to S exists and is denoted by $\mu^\sharp_S\mathcal{H}$. Moreover, by a tilting formula in [KQ15, Proposition 5.4], the new simples are $\operatorname{Sim}\mu^\sharp_S\mathcal{H}=\{S[1]\}\,\cup\,\{\psi^\sharp_S(X)\mid X\in\operatorname{Sim}\mathcal{H}, X\neq S\}$ where

$$\psi^\sharp_S(X) \;=\; \operatorname{Cone}\left(X\xrightarrow{f}S[1]\otimes\operatorname{Ext}^1(X,S)^*\right)[-1]\,. \tag{2.3}$$

Recall that the partial order on hearts $\mathcal{H}_1\le\mathcal{H}_2$ means $\mathcal{P}_1\supset\mathcal{P}_2\Leftrightarrow\mathcal{P}_1^\perp\supset\mathcal{P}_2^\perp$. The following is a characterization for all hearts in the interval $[\mathcal{H},\mathcal{H}[1]]$.

Lemma 2.1. *[KQ15, Remark 3.3] Fix a heart $\mathcal{H}$. Then a heart $\mathcal{H}'$ is a forward tilt of $\mathcal{H}$ if and only if $\mathcal{H}\le\mathcal{H}'\le\mathcal{H}[1]$. In this case, the tilting is with respect to the torsion pair $\mathcal{T}=\mathcal{H}'\cap\mathcal{H}$ and $\mathcal{F}=\mathcal{H}'[-1]\cap\mathcal{H}$.*

Stability structures. A *stability function* on an abelian category $\mathcal{H}$ is a group homomorphism $Z:K(\mathcal{H})\to\mathbb{C}$, such that for any $0\neq A\in\mathcal{H}$, $Z(A)\in\overline{\mathbb{H}}$. An object $A\in\mathcal{H}$ is said to be *Z-semistable* if for any non-zero proper sub-object $B\hookrightarrow A$ then $\frac{1}{\pi}\arg Z([B])\le\frac{1}{\pi}\arg Z([A])$. It is called *stable* if the inequality holds strictly. The quantity $\frac{1}{\pi}\arg Z(A)$ is called the *phase* of A. A stability function is called a *central charge* if it moreover satisfies the so-called *support property* and *Harder-Narasimhan property*, see [Bri07; KS08; BMS16] for more details.

A *stability condition* σ on $\mathcal{D}$ is a pair $\sigma=(Z,\mathcal{H})$, consisting on a heart $\mathcal{H}$ together with a central charge $Z\in\operatorname{Hom}(K(\mathcal{H}),\mathbb{C})$. Let

$$\mathcal{P}(\phi):=\{E[\lfloor\phi\rfloor]\mid E\text{ is }Z\text{-semistable in }\;\mathcal{H}\text{ of phase }\phi-\lfloor\phi\rfloor\}\;,\quad\forall\phi\in\mathbb{R}$$

be the *slice* consisting of semistable objects of phase ϕ. The collection of all slices is known as a slicing, denoted by $\mathcal{P}_{\mathbb{R}}:=\{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}}\subset\mathcal{D}$. Denote by $\mathcal{P}(I)=\langle P(\phi)\mid\phi\in I\rangle$ for any interval $I\subset\mathbb{R}$. The heart of a stability condition σ is $\mathcal{H}=\mathcal{P}(0,1]$, and the data of a stability conditions $\sigma=(Z,\mathcal{H})$ is equivalent to a pair $(Z,\mathcal{P}_{\mathbb{R}})$ with certain compatibility conditions, see [Bri07]. We recall the main result of [Bri07]:

Theorem 2.2. *The set of all stability conditions on $\mathcal{D}$ form a complex manifold* $\operatorname{Stab}(\mathcal{D})$ *with local coordinates given by the central charge $Z\in\operatorname{Hom}(K(\mathcal{D}),\mathbb{C})$.*

The group of autoequivalences $\operatorname{Aut}(\mathcal{D})$ acts on the left on $\operatorname{Stab}(\mathcal{D})$ by

$$\Phi.(Z,\mathcal{H})=\left(Z\circ\Phi^{-1},\Phi(\mathcal{H})\right),$$

which commutes with the action by scalars, for any $\lambda\in\mathbb{C}$:

$$\lambda(Z,\mathcal{P}) \;=\; (e^{-\pi i\lambda}Z,\mathcal{P}')\quad\text{where}\quad\mathcal{P}'(\phi)\;=\;\mathcal{P}(\phi+\operatorname{Re}\lambda). \tag{2.4}$$

Generic-finite connected components. We denote by $\mathrm{U}(\mathcal{H})$ the locus of stability conditions supported on a heart $\mathcal{H}$ and by $\mathrm{U}_0(\mathcal{H})$ its interior. If $\mathcal{H}$ is finite, then $\mathrm{U}(\mathcal{H}) = \overline{\mathbb{H}}^{|\operatorname{Sim}(\mathcal{H})|}$.

Definition 2.3. *Let $\mathcal{H}$ be a finite heart, and $\mathcal{W}_S(\mathcal{H}) \subset \mathrm{U}(\mathcal{H})$ be the real codimension 1 subset for which a simple S has phase 1 and all other simples in $\mathcal{H}$ have phase in $(0,1)$. We call $\mathcal{W}_S(\mathcal{H})$ a* wall. *A* chamber *is a connected component of the complement of the closure of the union of the walls in $\operatorname{Stab}(\mathcal{D})$.*

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite hearts, the intersection $\mathrm{U}(\mathcal{H}_1) \cap \overline{\mathrm{U}(\mathcal{H}_2)} = \mathcal{W}_S(\mathcal{H}_1)$ if and only if $\mathcal{H}_2 = \mu_S^\sharp \mathcal{H}_1$ by [Woo10]. We let

$$\operatorname{Stab}_0(\mathcal{D}) = \bigcup_{\mathcal{H}\text{ finite}} \mathrm{U}_0(\mathcal{H}) \quad \text{and} \quad \operatorname{Stab}_2(\mathcal{D}) = \bigcup_{\mathcal{H}\text{ finite}} \mathrm{U}(\mathcal{H}). \tag{2.5}$$

The indexing convention is parallel with the one that we will use for spaces of quadratic differentials in Section 4.2. If a connected component of $\operatorname{Stab}(\mathcal{D})$ has been specified, we decorate these spaces by a $\circ$ accordingly.

Definition 2.4. *A connected component $\operatorname{Stab}^\circ(\mathcal{D})$ is called*
- *a* finite type component, *if $\operatorname{Stab}^\circ \mathcal{D} = \operatorname{Stab}_2^\circ \mathcal{D}$;*
- *a* generic-finite type component, *if $\operatorname{Stab}^\circ \mathcal{D} = \mathbb{C} \cdot \operatorname{Stab}_2^\circ \mathcal{D}$.*

Abelian and triangulated quotient categories. Recall that a subcategory $\mathcal{S}$ of an abelian category $\mathcal{A}$ is called a *Serre subcategory* if it is abelian and for any short exact sequence $0 \to A_1 \to E \to A_2 \to 0$ in $\mathcal{A}$, we may conclude $E \in \mathcal{S}$ if and only if $A_1, A_2 \in \mathcal{S}$. In such a case, the quotient category $\mathcal{A}/\mathcal{S}$ is also abelian, cf. [Nee14, Lemma A.2.3] and [Gab62].

On the other hand, given a triangulated category $\mathcal{D}$ and a triangulated subcategory $\mathcal{V} \hookrightarrow \mathcal{D}$, we can construct the so-called *Verdier quotient* $\mathcal{D}/\mathcal{V}$. If $\mathcal{V}$ is thick (i.e. closed under direct summands), then

$$0 \to \mathcal{V} \to \mathcal{D} \to \mathcal{D}/\mathcal{V} \to 0$$

is a short exact sequence of triangulated categories with exact functors, [Ver96, Proposition 2.3.1].

Remark. *Whenever $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{V}$ is a quotient functor of triangulated categories, and $\mathcal{B}$ is a subcategory of $\mathcal{D}$, by $\pi(\mathcal{B})$ we will mean the* essential image *of $\mathcal{B}$ through π. This will apply in particular to the image of abelian hearts $\mathcal{H} \subset \mathcal{D}$.*

2.2. **Quivers with potential, mutation and Jacobian algebras.** In this article (Q, W) is a non-degenerate finite (possibly disconnected) oriented quiver $Q = (Q_0, Q_1, s, t)$ that has no loops or 2-cycles, with potential W considered up to right equivalence. The cyclic derivative of W with respect to an arrow $a \in Q_1$ is denoted $\partial_a W$. We refer to [DWZ08] for all these basic notions.

The operation of *mutation at a vertex* $i \in Q_0$, defined for instance in [KY11, Section 2], produces a new quiver with potential denoted $\mu_i(Q, W)$ or $(\mu_i Q, \mu_i W)$, that will have no loops nor 2-cycles.

The Jacobian algebra $\mathcal{J}(Q, W)$ of (Q, W) is the quotient of $\widehat{\mathbf{k}Q}$, the completion of the path algebra with respect to bilateral ideals generated by arrows, by the ideal $\partial W := \langle \partial_a W \mid a \in Q_1 \rangle$. The category of finite dimensional modules over $\mathcal{J}(Q, W)$ is denoted

$$\mathcal{H}(Q, W) := \operatorname{mod} \mathcal{J}(Q, W),$$

sometimes shortened to $\mathcal{H}_Q$ when the quiver with potential is clear. It is a finite length, finite, abelian category, see for instance [Kel11b, Section 3]. The vertices of the quiver give the simple objects of the module category $\mathrm{Sim}(\mathcal{H}_Q)$, so that in particular the Grothendieck group is $K(\mathcal{H}(Q,W)) = \mathbb{Z}^{|Q_0|}$.

Let $I \subset Q_0$ be a proper subset of the set of vertices of Q, and I^c its complement. There is an operation of *restriction* on (Q,W), to get a new quiver with potential, denoted by (Q_I, W_I) with vertices $(Q_I)_0 = I$. It is obtained by deleting vertices in I^c, arrows incoming or outgoing from vertices in I^c, and all cycles in W passing through vertices in I^c, [DWZ08]. We call (Q_I, W_I) a *(full) subquiver*. When $i \in I$, the operations of mutation μ_i and of restriction $|_I$ commutes, cf. [LF09]. The following is obvious.

Lemma 2.5. *Let $k \notin I \subset Q_0$ be a vertex of Q such that there are no arrows from i to k or from k to i for all $i \in I$. Then $(\mu_k(Q,W))_I = (Q_I, W_I)$.*

The finite-length property of $\mathcal{H}(Q,W)$ immediately implies:

Lemma 2.6. *There is a bijection between Serre subcategories of $\mathcal{H}(Q,W)$ and full sub-quivers (Q_I, W_I).*

We will be interested in the quotient abelian category

$$\mathcal{H}(Q,W)/\mathcal{H}(Q_I,W_I) \tag{2.6}$$

which is a category of modules over a finite dimensional algebra as well, [GL91, Propositions 2.2 and 5.3]. In particular it is a finite length, finite abelian category. The Grothendieck group splits as

$$K(\mathcal{H}(Q,W)) \simeq K(\mathcal{H}(Q_I,W_I)) \oplus K(\mathcal{H}(Q,W))/K(\mathcal{H}(Q_I,W_I)).$$

2.3. CY_3 **categories associated to a quiver with potential.** We denote by $\Gamma := \Gamma(Q,W)$ the *complete Ginzburg differential graded (dg) algebra* associated to a quiver with potential (Q,W). The underlying graded algebra Γ is the completion of the path algebra of a graded quiver obtained from Q and the differential is given by the potential W. It is defined in [Gin06; Kel06]. The zero-th homology of this algebra $H_0\big(\Gamma(Q,W)\big) \simeq \mathcal{J}(Q,W)$ gives back the Jacobian algebra of the original quiver with potential.

Recall that a category $\mathcal{C}$ is said to be Calabi-Yau of dimension N, or simply CY_N if for any objects $E, F \in \mathcal{C}$ there is a natural isomorphism $\nu : \mathrm{Hom}_{\mathcal{C}}(E,F) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(F, E[N])^\vee$ of $\mathbf{k}$-vector spaces.

Let $\mathcal{A}$ be a dg algebra with derived category $\mathcal{D}(\mathcal{A})$. Denote by $\mathrm{per}(\mathcal{A})$ and $\mathrm{pvd}(\mathcal{A})$ the *perfect derived category* and the *perfectly valued derived category* of $\mathcal{A}$, respectively. The perfect category $\mathrm{per}\,\Gamma$ is generated by the indecomposables projective dg modules $P_i = e_i\Gamma$, $i = 1, \ldots, n$. The perfectly valued derived category $\mathrm{pvd}(\Gamma)$ coincides with the subcategory of $\mathcal{D}(\Gamma)$ consisting on dg modules of total finite-dimensional homology.

We collect some well-known results from [Kel11a] and [KY11, Section 3-4].

Proposition 2.7. *For $\Gamma = \Gamma(Q,W)$ as above the following statements hold:*

- *The category $\mathrm{pvd}(\Gamma)$ is Hom-finite and CY_3, for any non-degenerated quiver with potential (Q,W), and it is contained in $\mathrm{per}(\Gamma)$.*
- *If $\Gamma' = \Gamma(Q',W')$ is obtained by mutation, then $\mathrm{pvd}(\Gamma') \cong \mathrm{pvd}(\Gamma)$ and $\mathrm{per}\,\Gamma' \cong \mathrm{per}\,\Gamma$.*

- *The category* $\operatorname{pvd}(\Gamma)$ *admits a standard heart of bounded t-structure*
$$\mathcal{H}(\Gamma) = \mathcal{H}(Q, W) := \operatorname{mod} \mathcal{J}(Q, W).$$

The CY_2 Verdier quotient $\operatorname{per}(\Gamma)/\operatorname{pvd}(\Gamma)$, sitting in a short exact sequence of triangulated categories,
$$0 \to \operatorname{pvd}(\Gamma) \to \operatorname{per}\Gamma \xrightarrow{\pi_\Gamma} \mathcal{C}(\Gamma) \to 0 \tag{2.7}$$
is called the *cluster category* and denoted by $\mathcal{C}(\Gamma)$, following Amiot [Ami09].

Let $(Q_I, W_I) = (Q, W)_I$ denote a full subquiver of (Q, W), as in the previous subsection, and $\Gamma_I = \Gamma(Q_I, W_I)$, $\mathcal{J}_I = \mathcal{J}(Q_I, W_I)$. The standard bounded t-structure $\mathcal{H}(\Gamma)$ on $\operatorname{pvd}(\Gamma)$ restricts to the standard bounded t-structure
$$\operatorname{mod} \mathcal{J}_I =: \mathcal{H}(\Gamma_I) = \operatorname{pvd}(\Gamma_I) \cap \mathcal{H}(\Gamma) \subset \mathcal{H}(\Gamma)$$
on the subcategory $\operatorname{pvd}(\Gamma_I) = \operatorname{thick}_{\operatorname{pvd}(\Gamma)} \mathcal{H}(\Gamma_I) \subset \operatorname{pvd}(\Gamma)$.

We are interested in the Verdier quotient $\operatorname{pvd}\Gamma/\operatorname{pvd}\Gamma_I$ and in those hearts that are the images, under the quotient functor, of a heart in $\operatorname{pvd}\Gamma$. We will study a component of the exchange graph of $\operatorname{pvd}\Gamma/\operatorname{pvd}\Gamma_I$ containing the heart $\mathcal{H}(\Gamma)/\mathcal{H}(\Gamma_I)$ in Section 6.

3. Decorated marked surfaces and quadratic differentials

3.1. **Quadratic differentials.** We set up notion for quadratic differentials, using the book of Strebel [Str84] as background.

Let X be a compact Riemann surface and ω_X be its holomorphic cotangent bundle. A *meromorphic quadratic differential* q on X is a meromorphic section of the line bundle ω_X^2. We denote by $\mathbf{z}$ the collection of points where q has a pole or vanishes, the *singularities* or *critical points* of q. These can be grouped into the finite critical points (zeros and simple poles) of q, and infinite critical poles (higher order poles). We denote by $Z_j(q)$ the set of finite critical points of q or order j and $P_k(q)$ the set of poles of q with order $k \geq 2$ Finally, let $Z(q) = \bigcup_{j \geq -1} Z_j(q)$ and $P(q) = \bigcup_{j \geq 2} P_j(q)$ and group them together as $\operatorname{Crit}(q) = Z(q) \cup P(q)$. We let $\mathbf{w} = (w_1, \ldots, w_r)$ be the orders of the finite critical points and $\mathbf{w}^- = (w_{r+1}, \ldots, w_{r+b})$ be the negative orders of higher order poles (i.e. $w_i \leq -2$ for $i \geq r+1$). The tuple $(\mathbf{w}, \mathbf{w}^-)$ is the *signature* of the quadratic differential.

The canonical covering construction. Associated with a quadratic differential q on a compact curve X there is a *canonical double cover* $\widehat{\pi} : \widehat{X} \to X$ such that $\widehat{\pi}^* q = \omega^2$ is the square of an abelian differential, unique up to sign. See e.g. [BCGGM1, Section 2.1] for various methods of construction. The tuple of preimages of the singularities of (X, q) is denoted by $\widehat{\mathbf{z}}$, and decomposed into the finite critical points $\widehat{Z}$ and infinite critical points $\widehat{P}$. To compute the signature of the double cover we define
$$(\widehat{\mathbf{w}}, \widehat{\mathbf{w}}^-) := \Big(\underbrace{\widehat{w}_1, \ldots, \widehat{w}_1}_{\gcd(2,w_1)}, \underbrace{\widehat{w}_2, \ldots, \widehat{w}_2}_{\gcd(2,w_2)}, \ldots, \underbrace{\widehat{w}_{r+b}, \ldots, \widehat{w}_{r+b}}_{\gcd(2,w_{r+b})}\Big), \tag{3.1}$$
where $\widehat{w}_i := \frac{2+w_i}{\gcd(2,w_i)} - 1$ and where now $\widehat{\mathbf{w}}^-$ is the tuple of the negative entries among these integers.

Trajectory structure. We now turn to the global trajectory structure of a quadratic differential q, following [Str84]. We suppose throughout that q has at least one zero and at least one *infinite critical point*, i.e. a pole of order ≤ -2 or equivalently that $\mathbf{w}^- \neq \emptyset$. We do not suppose that q has simple zeros (i.e. we do *not* work only with Gaiotto-Moore-Neitzke (GMN) differentials).

A *saddle connection* is a trajectory (in some arbitrary direction) whose maximal domain is a finite interval. Both its end points are zeros of q. A *saddle trajectory* is a saddle connection in the horizontal direction. A trajectory is *closed* if its a saddle trajectory and both its end points coincide. The remaining trajectories are either

(1) *separating*, i.e., approaching an infinite critical point at precisely one end,
(2) *recurrent* in at least one of its directions, or
(3) *generic*, approaching an infinite critical point in both directions.

We now fix the direction to be the horizontal direction unless specified otherwise, so 'trajectories' refers to 'horizontal trajectories'. Removing from X the separating trajectories and saddle trajectories decomposes the surface into connected components, which are of the following types.

(1) *ring domains* or *cylinders* that are foliated by closed trajectories,
(2) *horizontal strips* isometric to $S = \{a < \mathrm{Im}(z) < b\}$ with $q|_S = dz^{\otimes 2}$,
(3) *half-planes*, isometric to $\mathbb{H}$ with $q|_{\mathbb{H}} = dz^{\otimes 2}$, or
(4) *spiral domains*, the interior of the closure of a recurrent trajectory.

A ring domain is called *degenerate* if one of its boundary components is a double pole. A saddle trajectory is called *borderline* if it lies on the boundary of a degenerate ring domain, half-plane, or horizontal strip.

The quadratic differential q is called *saddle-free* if is does not have any saddle trajectories. By [BS15, Lemma 3.1] such a differential does have neither closed trajectories nor recurrent trajectories. In particular the complement of its saddle trajectories and separatrices is a union of half planes and horizontal strips. We call this the *horizontal strip decomposition* of (X, q).

Given a quadratic differential q on X we define the closed subsurface X^+ to be the closure of the union of all horizontal strips, half-planes and degenerate ring domains. The closed subsurface X^- is defined to be the closure of the union of all spiral domains and non-degenerate ring domains. The two subsurfaces $X^\pm$ meet along a collection of saddle connections, all of which are borderlines. See Figure 1 for examples.

If η is a path tracing a saddle connection on X, we let η' and η'' be the two lifts of the path to $\widehat{X}$. We define the lifted class $[\widehat{\eta}] \in H_1(\widehat{X} \setminus \widehat{P}, \widehat{Z}, \mathbb{Z})$ to be $[\widehat{\eta}] = [\eta']$ if $[\eta'] + [\eta''] = 0 \in H_1(\widehat{X} \setminus \widehat{P}, \widehat{Z}, \mathbb{Z})$ and we define $[\widehat{\eta}] = [\eta'] - [\eta'']$ otherwise. We declare two saddle connections η_1 and η_2 to be *hat-homologous* if for some choice of orientation the equality $[\widehat{\eta}_1] = [\widehat{\eta}_2]$ holds in $H_1(\widehat{X} \setminus \widehat{P}, \widehat{Z}, \mathbb{Z})$. We say that two saddle connections are *hat-proportional* if $[\widehat{\eta}_1]$ and $[\widehat{\eta}_2]$ are proportional. The characterization in [MZ08, Proposition 1] via rigid configurations shows that saddle connections are hat-proportional if and only if they are hat-homologous. This is the reason for our definition of $[\widehat{\eta}]$, which differs sometimes by a factor 2 from the one in [BS15], compare with [Ike17].

3.2. **Decorated marked surfaces.** The notion of marked surface encodes the raw combinatorics of a quadratic differential with the limit points of trajectories at the poles and possible additional auxiliary punctures, but without specifying order and location of the zeros. Marked points are usually referred to as *prongs* at the poles in flat surface literature. Here we follow [BS15, Section 3] and [KQ20, Section 4] to relate quadratic differentials and weighted marked surfaces.

Definition 3.1. *A* marked surface $\mathbf{S} = (\mathbf{S}, \mathbb{M}, \mathbb{P})$ *consists of a connected bordered differentiable surface with a fixed orientation, together with a finite set* $\mathbb{M} = \bigcup_{i=1}^{b} M_i$ *of marked point on the boundary* $\partial\mathbf{S} = \bigcup_{i=1}^{b} \partial_i$ *and a finite set* $\mathbb{P} = \{p_j\}_{j=1}^{p}$ *of punctures in its interior* $\mathbf{S}^\circ = \mathbf{S} - \partial\mathbf{S}$*, such that each connected component of* $\partial\mathbf{S}$ *contains at least one marked point.*

Up to homeomorphism, $\mathbf{S}$ is determined by the following data

- the genus g;
- the number b of boundary components;
- the number $p = \#\mathbb{P}$ of punctures;
- the negative integer partition $\mathbf{w}^-$ of $-m = -\#\mathbb{M}$ into b parts describing the number of marked points on its boundary, and consisting in $w_i^- = -\#\mathbb{M}_i$.

The rank of $\mathbf{S}$ is defined to be

$$N \;=\; 6g + 3p + 3b + m - 6. \tag{3.2}$$

For simplicity, we only consider the $\mathbb{P} = \emptyset$ *case in this paper.*

Decorations and weight add to a marked surface the data of the location and orders of zeros of a differential.

Definition 3.2. *A* decorated marked surface *(abbreviated as* DMS*) is obtained from a marked surface* $\mathbf{S}$ *by decorating it with a set* $\Delta = \{z_i\}_{i=1}^{r}$ *of points in the surface interior* $\mathbf{S}^\circ$*. These points are called* finite critical points*. A* weight *function on* Δ *is a* $\mathbb{Z}_{\geq -1}$*-valued function*

$$\mathbf{w}\colon \Delta \to \mathbb{Z}_{\geq -1}\,.$$

We write $r = |\mathbf{w}| = |\Delta|$ *for the number of finite critical points and* $\|\mathbf{w}\| = \sum_{Z\in\Delta} \mathbf{w}(Z)$ *for their total weight. We say* $\mathbf{w}$ *is* compatible *with* $\mathbf{S}$ *if*

$$\|\mathbf{w}(Z)\| - (m + 2b) \;=\; 4g - 4\,. \tag{3.3}$$

If $\mathbf{w}$ *and* $\mathbf{S}$ *is compatible, we will write* $\mathbf{S}_{\mathbf{w}} = (\mathbf{S}, \Delta, \mathbf{w})$ *and call this tuple a weighted DMS (abbreviated as wDMS).*

A weight $\mathbf{w}$ is *simple* if $\mathbf{w} \equiv 1$. We write $\mathbf{S}_\Delta$ to indicate that we work with a wDMS with simple weight. This is the case studied previously, e.g., in [Qiu16; Qiu18; QZ20; BQZ21; KQ20], and corresponds to the setting of principal strata of quadratic differentials discussed in [BS15].

Quadratic differentials on marked surfaces. Fix a quadratic differential q and a let $\theta \in S^1$ be a direction. A maximal straight arc (for the metric $|q|$) in the direction θ is called *trajectory* (in the direction θ). Locally near a finite critical point of order $w \geq -1$ there are $w + 2$ distinguished directions that are limits of a trajectory in the direction θ. Similarly, at a pole p of order $|w| := \mathrm{ord}_q(p) \geq 3$ there are $|w| - 2$ distinguished directions that are limits of a trajectory in the direction θ. These directions are called *prongs* at the zero or pole.

The *real (oriented) blow-up* of (X, q) is the differentiable surface X^q, which is obtained from X by replacing a pole $p \in P(q)$ of order at least 3 by a boundary circle $\partial_p \cong S^1$. Moreover, we mark the points on ∂_p that correspond to the distinguished tangent directions, so there are $\mathrm{ord}_q(p) - 2$ marked points on ∂_p. This turns X^q into a marked surface. Adding the set of zeros $Z(q)$ together with their orders as weight make X^q into a wDMS, the *weighted decorated real blow-up of* (X, q). Fixing moreover a diffeomorphism to a reference surface gives a framing of (X, q).

Definition 3.3. *Fix a wDMS* $\mathbf{S_w}$*. An* $\mathbf{S_w}$-framed quadratic differential (X, q, ψ) *is a Riemann surface* X *with quadratic differential* q*, equipped with a diffeomorphism* $\psi\colon \mathbf{S_w} \to \mathbf{X}^q$*, preserving the marked points, decorations and their weights.*

Two $\mathbf{S_w}$*-framed quadratic differentials* (X_1, q_1, ψ_1) *and* (X_2, q_2, ψ_2) *are isomorphic, if there exists a biholomorphism* $f\colon X_1 \to X_2$ *such that* $f^*(q_2) = q_1$ *and furthermore* $\psi_2^{-1} \circ f_* \circ \psi_1 \in \mathrm{Diff}_0(\mathbf{S_w})$ *is isotopic to the identity preserving marked points, decorations and their weights (setwise). Here* $f_*\colon (\mathbf{X}_1)^{q_1} \to (\mathbf{X}_2)^{q_2}$ *is the induced diffeomorphism on real oriented blowups.*

In flat surface literature this kind of framing is usually called a (Teichmüller) marking. To avoid confusion with the (prong) markings used here, we stick to the terminology common to e.g. [BS15] and [KQ20], but we use "Teichmüller" to refer to this kind of marking without specifying the underlying wDMS.

3.3. **Arc systems.** We consider a decorated marked surfaces (wDMS) $\mathbf{S_w}$ with decorations Δ, weight $\mathbf{w} : \Delta \to \mathbb{Z}_{\geq -1}$, and marked points $\mathbb{M}$. We let $\mathbf{S}^\circ_\mathbf{w} := \mathbf{S_w} \backslash \partial \mathbf{S_w}$ and introduce the following additional notation.

- An *open arc* is an (isotopy class of a) curve $\gamma\colon I \to \mathbf{S_w}$ such that its interior is in $\mathbf{S}^\circ_\mathbf{w} \setminus \Delta$ and its endpoints are in the set of marked points $\mathbb{M}$.
- A *closed arc* is a curve $\eta\colon I \to \mathbf{S_w}$ such that its interior is in $\mathbf{S}^\circ_\mathbf{w} \setminus \Delta$ and its endpoints are in the set of decoration points Δ. (To memorize: The interval that maps to $\mathbf{S}^\circ_\mathbf{w}$ is *closed*.)

For the simply decorated case, i.e. for $\mathbf{w} \equiv \mathbf{1}$, we denote by $\mathrm{CA}(\mathbf{S}_\Delta)$ the set of closed arcs on $\mathbf{S}_\Delta = \mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$ that have no self-intersections, not even at the endpoints in Δ. Similarly, let $\mathrm{OA}(\mathbf{S}_\Delta)$ be the set of open arcs of $\mathbf{S}_\Delta$. Throughout this paper γ's denote open arcs and η's denote closed arcs, unless stated otherwise.

An *(open) arc system* $\mathbb{A} = \{\gamma_i\}$ is a collection of open arcs on $\mathbf{S_w}$ such that there is no (self-)intersection between any of them in $\mathbf{S}^\circ_\mathbf{w}$. Open arc systems first appeared for triangulations of simply weighted marked surfaces (i.e., $\mathbf{w} \equiv \mathbf{1}$). A *triangulation* $\mathbb{T}$ of $\mathbf{S}_\Delta$ is a maximal arc system of open arcs, which in fact divide $\mathbf{S}_\Delta$ into triangles. Two triangulations are related by a *flip* if they only differ by one arc. Locally, the two arcs involved in a flip are the two diagonals of a square.

We now move on to the weighted version of this notion. The motivation for the notion is provided in Section 4.3, compare also with [Kra08].

Definition 3.4. *A* $\mathbf{w}$-mixed-angulation *of* $\mathbf{S_w}$ *is a collection of open arcs that divides* $\mathbf{S_w}$ *into once-decorated polygons, such that each decoration* z *with weight* $w = w(z)$ *is in a* $(w+2)$*-gon. We denote this* $(w+2)$*-gon by* $\mathbb{A}(z)$ *and call it an* $\mathbb{A}$-polygon*.*

The forward flip *on* $\mathbf{w}$*-mixed-angulation* $\mathbb{A}$*, with respect to an arc* $\gamma \in \mathbb{A}$*, is an operation that moves the endpoints of* γ *anti-clockwise along two adjacent sides of the* $\mathbb{A}$*-polygons containing* γ*, cf. Figure 1.*

Although the definition allows for 1-gon and 2-gons, we will consider decorations with weight at least one only (i.e. k-gons with $k \geq 3$) in accordance with the standing assumption from the introduction and the one in Section 5.1 below.

When $\mathbf{S_w} = \mathbf{S}_\Delta$ has simple weight $\mathbf{w} \equiv \mathbf{1}$, the $\mathbf{w}$-mixed-angulations are (decorated) triangulations of $\mathbf{S}_\Delta$. We recall here that a quiver $Q_\mathbb{T}$ (without loops or

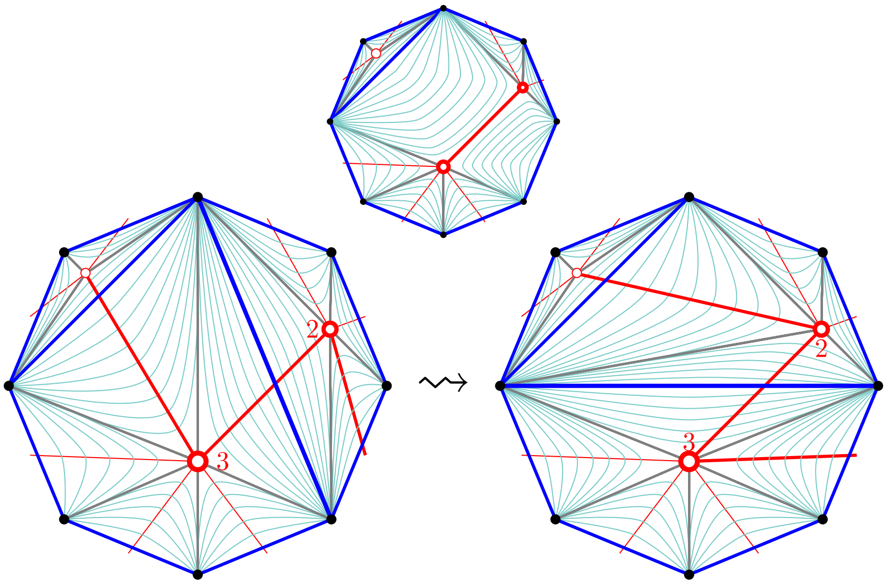

Figure 1. The figure shows several local horizontal strip decompositions on $\mathbf{S}_{\mathbf{w}}$ with fixed weighted decorations, depending on a quadratic differential q. Here the black vertices are marked points on $\partial \mathbf{S}_{\mathbf{w}}$, the red vertices are weighted zeros of q, the green arcs are geodesics, the black arcs are separating trajectories. The blue lines define $\mathbf{w}$-mixed-angulations of $\mathbf{S}_{\mathbf{w}}$. The red solid arcs are simple saddle connections, (except for the thick one in the top small octagon, which is a saddle trajectory) and represent the duel graphs of the $\mathbf{w}$-mixed-angulations. The picture in the middle represents crossing a wall of second kind, resulting in a forward flip.

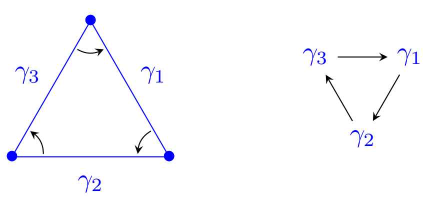

Figure 2. Local 3-cycle associated to a triangle of $\mathbb{T}$

2-cycles) with a potential $W_{\mathbb{T}}$ can be associated to a triangulation $\mathbb{T}$ of a simply decorated marked surface as follows:

- the vertices of $Q_{\mathbb{T}}$ correspond to the open arcs in $\mathbb{T}$;
- the arrows of $Q_{\mathbb{T}}$ correspond to (anticlockwise) oriented intersection between open arcs in $T\mathbb{T}$, so that there is a 3-cycle in $Q_{\mathbb{T}}$ locally in each triangle as shown in Figure 2.
- the potential $W_{\mathbb{T}}$ is the sum of all 3-cycles that locally coming from a triangle of $\mathbb{T}$ as above.

The corresponding Ginzburg algebra $\Gamma(Q_{\mathbb{T}}, W_{\mathbb{T}})$ will usually be denoted by $\Gamma_{\mathbb{T}}$.

4. Moduli spaces of quadratic differentials

The main goal is the proof that "walls have ends" in Proposition 4.2, which is used in Corollary 4.3 to homotope paths into the locus of so-called tame differentials, inside the moduli space of quadratic differentials. The proof of our main result, Theorem 7.2, relies on this corollary. These two results are generalization of results in [BS15] where those statements are proven for differentials with simple zeros only. Their proof relies crucially on this hypothesis in [BS15, Lemma 5.1]. Our proof avoids most of the discussion of configuration of hat-homologous saddle connections and uses more elaborate ways to deform half-translation surfaces instead. We start by recalling some moduli spaces of quadratic differentials.

4.1. **Space of quadratic differentials.** Following flat surface literature we let $\mathrm{Quad}_{g,r+b}(\mathbf{w},\mathbf{w}^-)$ be the moduli space of quadratic differentials $(X,\mathbf{z},q)$ on a pointed curve $(X,\mathbf{z})$ where $\mathbf{z}=(z_1,\dots,z_{r+b})$ such that q has signature $(\mathbf{w},\mathbf{w}^-)$. We emphasize that in this space the critical points are labeled. The unlabeled version is denoted by $\mathrm{Quad}_g(\mathbf{w},\mathbf{w}^-)$, i.e., without the subscript for the number of labeled points. Since every quadratic differential is compatible with a (unique, up to diffeomorphism) wDMS $\mathbf{S_w}$, which encodes both the zeros (via weight) and the polar part of the signature (via the marking), we also use the notation $\mathrm{Quad}(\mathbf{S_w}) = \mathrm{Quad}_g(\mathbf{w},\mathbf{w}^-)$. Next we discuss several types of framed moduli spaces. Note that $\mathrm{Quad}(\mathbf{S_w})$ is in general an orbifold and non-connected. The (finite) number of components is classified in some cases in [CG22].

Framings by periods. Spaces of quadratic differentials are locally modeled on the anti-invariant eigenspace of the relative cohomology of the canonical cover, the so-called *hat-cohomology*. We fix a quadratic differential q of signature $(\mathbf{w},\mathbf{w}^-)$ on a surface X and recall that $\widehat{Z}$ and $\widehat{P}$ are preimages of zeroes and poles on the double cover. Then the *hat-homology group* with integral coefficients is defined as

$$\Gamma := \widehat{H}_1(q) \;=\; H_1(\widehat{X}\setminus\widehat{P},\widehat{Z},\mathbb{C})^-\,, \tag{4.1}$$

where the minus sign denotes the antiinvariant part of the homology with respect to the involution τ whose quotient map is the canonical double cover $\pi:\widehat{X}\to X$.

Period coordinates, i.e. integrating the one-form on the double cover against a basis of hat-homology, give a local isomorphism

$$\mathrm{Per}: U(q)\to H^1(\widehat{X}\setminus\widehat{P},\widehat{Z},\mathbb{Z})^- \;=\; \mathrm{Hom}(\Gamma,\mathbb{C}) \tag{4.2}$$

on a neighborhood $U(q)$ of q in the moduli space of quadratic differentials. Note that if all the entries of $\mathbf{w}$ are odd, the hat-homology group is unchanged if we do not consider homology relative to the zeros. ('The principal strata of quadratic differentials have no REL'.)

To globalize the period map we fix a trivialization of the hat-homology group. That is, we fix a reference differential (X_0,q_0) and define

$$\mathrm{Quad}^\Gamma_g(\mathbf{w},\mathbf{w}^-) \;=\; \{(X,q,\rho)\in\mathrm{Quad}_g(\mathbf{w},\mathbf{w}^-),\quad \rho:\widehat{H}_1(q_0)\xrightarrow{\cong}\widehat{H}_1(q)\}\,,$$

the space of *period-framed quadratic differentials* of signature $(\mathbf{w},\mathbf{w}^-)$.

(Teichmüller) framed quadratic differentials. For fixed discrete data $(g, b, \mathbf{w}^-, \mathbf{w})$ we denote by $\mathrm{FQuad}(\mathbf{S_w})$ the moduli space of $\mathbf{S_w}$-framed quadratic differentials. This moduli space is a manifold, but non-connected. We denote by $\mathrm{FQuad}^\circ(\mathbf{S_w})$[1] a connected component, in applications typically singled out to contain a given $\mathbf{S_w}$-framed differential.

These spaces are strata of a vector bundle. The top dimensional stratum of this vector bundle is $\mathrm{FQuad}(\mathbf{S}_\Delta)$ with simple weighted decorations.

Mapping class group action. In our context two mapping class groups are important. In general, the *(unpunctured) mapping class group of a marked surface* $\mathbf{S}$ is the group $\mathrm{MCG}(\mathbf{S})$ of isotopy classes of diffeomorphisms of $\mathbf{S}$ relative to the boundary and marked points. Similarly we define the *full mapping class group* $\mathrm{MCG}(\mathbf{S_w}) = \mathrm{MCG}(\mathbf{w}, \mathbf{w}^-)$ as diffeomorphisms with the additional condition to respect finite critical points and their weight (set-wise).

The mapping class group acts on the set of all Teichmüller framings by precomposition. Obviously $\mathrm{Quad}(\mathbf{S_w}) = \mathrm{FQuad}(\mathbf{S_w})/\,\mathrm{MCG}(\mathbf{S_w})$ as orbifolds.

4.2. **Walls have ends and homotopies to tame paths.** In our case, just as for the GMN-differentials treated in [BS15, Section 5], the space $\mathrm{Quad}(\mathbf{S_w})$ has a stratification by the number of horizontal saddle connections. The difference is that the number of horizontal trajectories emerging from a zero is not three, but $w_i + 2$ if the zero is of order w_i. This means that the number s_q of saddle trajectories, the r_q recurrent trajectories and the number t_q of separating trajectories satisfy

$$k \;:=\; r_q + 2s_q + t_q \;=\; \sum_{i=1}^{r_1} (w_i + 2)\,. \tag{4.3}$$

Stratification. We define

$$B_p \;\coloneqq\; B_p(\mathbf{S_w}) \;=\; \{q \in \mathrm{Quad}(\mathbf{S_w}) : r_q + 2s_q \le p\} \tag{4.4}$$

and observe that $B_0 = B_1$ is the set of saddle-free differentials by the preceding observation. There is an increasing chain of subspaces

$$B_0 = B_1 \subset B_2 \subset \cdots \subset B_k \;=\; \mathrm{Quad}(\mathbf{S_w})\,. \tag{4.5}$$

This follows from the lower semicontinuity of the function t_q on X. The space B_2 is called the space of *tame differentials*. We define the stratification

$$F_p \;\coloneqq\; F_p(\mathbf{S_w}) \;=\; B_p \setminus B_{p-1}. \tag{4.6}$$

We observe that F_0 is dense, F_1 is empty, and F_2 consists of differentials with exactly one saddle trajectory, since the boundary of a spiral domain has a saddle trajectory ([BS15, Lemma 3.1]). In fact, we have the more precise statement from [BS15, Lemma 4.11] and [Aul18, Theorem 1.4].

Lemma 4.1. *$B_0(\mathbf{S_w})$ is dense in $\mathrm{FQuad}(\mathbf{S_w})$. In fact, $\mathrm{FQuad}(\mathbf{S_w}) = \mathbb{C} \cdot B_0(\mathbf{S_w})$.*

We will see that B_p is not always locally finite, and even if it is, the relation between the integer p and the codimension of B_p is complicated and depends on s_q and the geometry of the spiral domains.

[1] The $\circ$ should remind of the symbol for the connected component of the identity in a topological group.

We can now state and prove our goal, the generalization of [BS15, Proposition 5.8] to zeros of arbitrary order. It follows as a corollary of the following Proposition 4.2 which is proven in the sequel.

Proposition 4.2. *Suppose that $p > 2$ and suppose that the negative part of the signature is not $\mathbf{w}^- = (-2)$. Then each component of the stratum F_p contains a point q and a neighborhood $U \subset \mathrm{Quad}_g(\mathbf{w}, \mathbf{w}^-)$ of q such that $U \cap B_p$ is contained in the locus $\mathrm{Per}(\alpha) \in \mathbb{R}$ for some $\alpha \in \Gamma$, and that this containment is strict in the more precise sense that $U \cap B_{p-1}$ is connected.*

The locus $\mathrm{Per}(\alpha) \in \mathbb{R}$ appearing in the first property is the wall (i.e., a real codimension one locus) the subsection title alludes to, and the second statement guarantees the end of this wall. We will apply this proposition in the following form:

Corollary 4.3 ([BS15, Proposition 5.8])**.** *Suppose that the negative part of the signature is not $\mathbf{w}^- = (-2)$. Then any path in $\mathrm{Quad}_g(\mathbf{w}, \mathbf{w}^-)$ can be homotoped relative to its end points to a path in B_2.*

Sketch of proof. Suppose the path lies in B_p. We inductively reduce p by first perturbing it so that it intersects F_p in only finitely many points. For each of them, drag the path along the nearby B_{p-1} to an end of the wall given by Proposition 4.2, go around and return to the other side of the intersection point with the wall. ☐

Lemma 4.4. *Let R be a ring domain in a surface (X, q) that belongs to a stratum $F_p \subset \mathrm{Quad}_g(\mathbf{w}, \mathbf{w}^-)$ and let $X^c = X \setminus \overline{R}$ be the complement of the closed ring domain. Then there exists a path $\alpha : [0, 1] \to F_p$ such that $\alpha(0) = (X, q)$, such that X^c is unchanged along α and such that $\alpha(1) = \overline{X^c}$ is the closure of the ring domain complement.*

Proof. Let I be the intersection $\overline{X}^c \cap \overline{R}$ of the ring domain with the rest of the surface. Let β be a saddle connection crossing the ring domain once. Consider horizontal twists of the cylinder R, i.e. the action of the upper triangular group on R while not changing X^c. This changes the period of β by some real number while keeping the lengths of all saddle trajectories fixed. We choose this twist so that there is no vertical saddle connection emanating from a zero on ∂R that stays within $\overline{R}$. (The set of twists where such a vertical saddle connection does exist is countable.) This is the first part of the path α.

Now we shrink the height of the cylinder, i.e., the imaginary part of the period of β to zero. We claim that we stay in $\mathrm{Quad}_g(\mathbf{w}, \mathbf{w}^-)$ during this process. This is proven in detail in [AW21, Section 4.3] and sketched in [MW17, Section 3.1]. The idea is to draw the vertical separatricies in the cylinder until they leave the cylinder. This has to happen, since otherwise we'd have a vertical spiral domain, the boundary of which has vertical saddle connections, but we excluded these. These vertical lines divide the cylinder into rectangles. In the limiting surface at $\mathrm{Im}(\beta) = 0$ the top and bottom of each of these rectangles (considered inside the surface $\overline{X}^c$ slit open along I) are glued together.

To see that this path stays in F_p note that all union of the rays emanating into I and on the boundary of R are saddle trajectories for each surface along the path p just described, including its end points. Since the set of this rays is constant along p and since X^c is unchanged along the path, the claim follows. ☐

We can also get rid of spiral domains by small perturbation in a fixed stratum. Recall from [Str84, Section 11.2] that the boundary of a spiral domain consists of saddle trajectories.

Lemma 4.5. *Let S be a spiral domain in a surface (X,q) that belongs to a stratum $F_p \subset \mathrm{Quad}_g(\mathbf{w},\mathbf{w}^-)$ and let $X^c = X \setminus \overline{S}$ be the complement of the closure of the spiral domain. Then there exists a path $\alpha : [0,1] \to F_p$ such that $\alpha(0) = (X,q)$, such that X^c is unchanged along α and such that $\alpha(1) \setminus X^c$ contains a ring domain.*

Proof. Since S is a spiral domain there is at least one saddle connection β starting in the interior of the spiral domain $\overline{S}$ with $\mathrm{Per}(\beta) \notin \mathbb{R}$, say oriented to have positive imaginary part. To show this we can e.g. use the decomposition of the spiral domain into rectangles from [Str84, Section 11.3]: if there was no zero in the interior, this decomposition would exhibit the spiral domain actually as a ring domain. An arbitrarily small purely imaginary deformation of β will create a saddle trajectory that intersects $\overline{X}^c$ at most at its end points. Since we may make the deformation smaller than the shortest saddle connections, no two points have collided and we stay in the space $\mathrm{Quad}_g(\mathbf{w},\mathbf{w}^-)$. If after this deformation the complement of X^c does not yet contain a ring domain it must contain spiral domains and we can repeat the procedure, creating a new saddle trajectory at each step. The process has to terminate once $p = 2s_q$ and then $r_q = 0$, i.e. the complement of X^c must contain a ring domain.

We argue that we stay in F_p along this process. This follows since X^c is unchanged in the whole process, and since all horizontal trajectories emanating from a zero into the complement of X^c contribute to $r_q + 2s_q$ at any stage of the process. □

Proof of Proposition 4.2. The beginning of the following proof follows [BS15, Proposition 5.8], replacing an argument using generic (in the sense of loc. cit.) differentials by an alternative argument. The second part is based on our version of the surface perturbations.

First recall the following Lemma due to [BS15], that provides the end of the wall, if the η_i are independent in hat-homology so that their periods can be modified independently.

Lemma 4.6 ([BS15, Proposition 5.3])**.** *Suppose that q_0 has a half-plane or a horizontal strip bounded by exactly s saddle trajectories γ_i, numbered consecutively. Let $\alpha = \sum_{i=1}^{s} \gamma_i$. Then there is an open neighborhood U of q_0 such that*

$$\text{if } q \in U \cap F_p \quad \text{then} \quad \mathrm{Per}(\alpha) \in \mathbb{R}\,.$$

Moreover, $q \in U \cap F_p$ implies that

$$\mathrm{Im}\Big(\sum_{i=1}^{k} \mathrm{Per}(\gamma_i)\Big) \leq 0 \tag{4.7}$$

for all $0 < k < s$, if the surfaces is oriented such that a half-plane or a horizontal strip is above the real axis.

Proof of Proposition 4.2. Consider any point $q \in F_p$ with $p > 2$. In this situation there is a borderline saddle connection. Hence for a sufficiently small neighborhood U we have

$$U \cap F_p \subseteq \{q : \mathrm{Per}(\alpha) \in \mathbb{R}\} \tag{4.8}$$

for some $\alpha \in \Gamma$ by Lemma 4.6 and the analogous [BS15, Lemma 5.4] for the boundary of a degenerate ring-domain. A neighborhood U satisfying the first property is thus available for every q.

Suppose that q has only one saddle trajectory. Then, since $p > 2$, there must exist a spiral domain and X^- must be non-empty. The intersection $X^+ \cap X^-$ thus consists of one saddle trajectory only. This saddle trajectory has to be the boundary of a degenerate ring domain, and since any component of X^+ contains an infinite critical point and a saddle trajectory on its boundary, there is only one double pole, contradicting the hypothesis.

Consequently, we may assume that there are at least two saddle trajectories. More precisely, we may assume that α is the class of a union of saddle trajectories on the boundary of one component of X^+ and that either there is another component of X^+ with boundary class α', or that $s \geq 2$ in Lemma 4.6.

If the inclusion in (4.8) is strict, we are done. This happens if $s \geq 2$ by (4.7), if moreover not all γ_i in this lemma are hat-proportional and so two of them can be moved independently. This also happens if α' and α are not hat-proportional (by tilting α'), or equivalently if they are hat-homologous.

We thus need to analyze the situation that q has two or more borderline saddle trajectories $\gamma_1, \gamma_2, \ldots$ and all the borderline saddle trajectories are hat-proportional. If a single γ_i or a union of these separates off a subsurface X_0 contained in X^-, i.e. without poles, then we are done by the following *subsurface argument*: As long as the subsurface contains spiral domains we apply Lemma 4.4, creating a new cylinder each time. Since the number of horizontal cylinders is bounded by the topology, this procedure terminates. Now we apply successively Lemma 4.5 to each of these cylinders. Note that a saddle connection crossing a cylinder cannot be hat-homologous to γ_i. Consequently we arrive after the ring domain shrinking process at a point where we conclude by Lemma 4.6.

In general there are three cases depending on the position of the first two, say, of these trajectories γ_i.

Case 1: Suppose both of them are closed. If there is a path starting and ending at a pole crossing one γ_i but not the other, then the two are not hat-proportional, since the Lefschetz pairing (see [Spa66, Theorem 6.2.17])

$$H_1(\widehat{X} \setminus \widehat{P}, \widehat{Z}, \mathbb{Z}) \times H_1(\widehat{X} \setminus \widehat{Z}, \widehat{P}, \mathbb{Z}) \to \mathbb{Z}$$

is non-degenerate. The only case not yet covered by the subsurface argument is that γ_1 and γ_2 jointly cut X into two components, one of which has no higher order poles, i.e. belongs to X^-. We conclude again by the subsurface argument applied to the component without higher order poles.

Case 2: Suppose none of them is closed. If $\gamma_1 \cup \gamma_2$ does not separate the surface, take a path joining a pole to itself, crossing γ_1 once, but not γ_2. Take one of the lifts of this path to the canonical cover and use that Lefschetz pairing to obtain a contradiction to $[\widehat{\gamma}_1] = [\widehat{\gamma}_2]$ in hat-homology. If there are poles on both sides of this loop, the same Lefschetz pairing argument applies. It remains to deal with the case that γ splits off a subsurface in X^-, which is being dealt with by the subsurface argument.

Case 3: Suppose that precisely one of them, say γ_1, is closed. As in Case 1, if γ_1 separates off a surface without poles we conclude by the subsurface argument. Otherwise there is a path starting and ending at a pole, crossing γ_1 once and not crossing γ_2. The lift of this path to $\widehat{X}$ and the Lefschetz pairing invalidates that γ_1 and γ_2 are hat-proportional in $H_1(\widehat{X} \setminus \widehat{P}, \mathbb{Z})$. $\square$

The locus B_2 is not locally connected. We show that in general the homotopy to a tame path can not be performed locally. Consider an elliptic curve whose horizontal leaves are dense. Make a slit and glue the two sides of the slit (one after rotation by π) to adjacent saddle trajectories on the top of a half-plane. This results in a surface in $\mathrm{Quad}_1(2, 1, -3)$, consisting of a spiral domain and the half plane. The two slit segments γ_1, γ_2 are hat-homologous. This type of surfaces belongs to F_6, and B_6 has locally $\mathbb{R}$-codimension one, cut out by $\mathrm{Per}(\gamma_1) \in \mathbb{R}$.

4.3. **Mixed-angulations from quadratic differentials.** This section gives the geometric justification for introducing $\mathbf{w}$-mixed angulations by studying quadratic differentials with higher order poles, and shows that adjacency of chambers of saddle-free quadratic differentials is encoded by flips of mixed-angulations.

Definition 4.7. *Let $(X, q, \psi\colon \mathbf{S_w} \to X^q)$ be an $\mathbf{S_w}$-framed quadratic differential which is saddle-free. Then there is a $\mathbf{w}$-mixed-angulation $\mathbb{A}_q$ on $\mathbf{S_w}$* induced from q *(or more precisely from (q, ψ)) where the open arcs are inherited from (isotopy classes of) generic trajectories.*

The dual graph $\mathbb{A}_q^*$ also has a geometric interpretation. Its arcs represent the saddle connections crossing once each horizontal strip. It can be enhanced with a ribbon-graph structure and as such carries the information about $\mathbf{w}$. We refer to $\mathbb{A}_q^*$ as the $\mathbf{w}$-*ribbon graph induced by q*. The trajectory structure on $\mathbf{S_w}$ induced by a quadratic differential, hence the local picture of $\mathbb{A}_q$ and its dual are illustrated together with the effect of a forward flip in Figure 1.

Definition 4.7 implies that each component of the locus $B_0 \subset \mathrm{FQuad}(\mathbf{w}, \mathbf{w}^-)$ of saddle-free differentials gives the same mixed-angulation. We next highlight the role of the locus of tame differentials:

Proposition 4.8. *Two components of B_0 can be connected by an arc in B_2 with only one point non-saddle-free if and only if the corresponding $\mathbf{w}$-mixed-angulations are related by a forward flip.*

Proof. Suppose that the two components of B_0 are connected by such an arc, which we may homotope to be a small rotation of a saddle connection near the real axis while fixing the geometry of the rest of the surface. The question is thus local, in the neighborhood of this saddle connection. Using a metrically correct drawing, as in the middle of Figure 3 one checks that rotating in clockwise (anticlockwise) direction has the effect of passing from the leftmost to the rightmost picture in terms of horizontal strip decompositions. Picking a generic trajectory from the strips, we observe that this changes the mixed-angulation by a forward flip (backward flip).

Conversely, if two mixed-angulations differ by a forward flip we take differentials locally as indicated in the metric picture and rotate the saddle connection to produce a path as required. $\square$

We will recast this statement in terms of exchange graphs and generalize it to collapsed surfaces in Section 5.

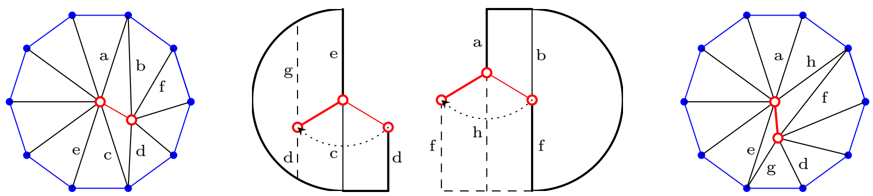

FIGURE 3. Horizontal foliation before and after rotating

5. Subsurface collapsing

In this section we formalize in the notion of *collapse of a subsurface.* In the special case of collisions, just a collection of simply decorated points (but no topology) are pinched. This will be the simplest ways to realize the generalized Bridgeland-Smith correspondence, but also the general case will play a role in sequels.

We summarize several notions of exchange graphs, related to tilting, mutations and flips, and recall the relations between them, thereby introducing spherical twist groups and braid twist groups. In particular we recall an isomorphism between exchange graphs for $\mathrm{pvd}(\Gamma)$ and for decorated marked surfaces with simple weights. This isomorphism will subsequently be generalized to non-simple weights. As preparation on the topological side we analyze refinements of mixed-angulations. Finally, we show auxiliary connectivity results for the graph of refinements to be used in the next section.

5.1. **Collapse of subsurfaces.** Let Σ be a subsurface of a weighted DMS $\mathbf{S}_{\mathbf{w}^0}$, possibly disconnected with connected components Σ_i. We denote by c_{ij} the (simple closed) curves such that the union $\cup_j c_{ij}$ forms the intersection of the boundary of Σ_i with the boundary of $\mathbf{S}_{\mathbf{w}^0} \setminus \Sigma$. These will be the boundary components of Σ we will be most interested in. An assignment of integers κ_{ij} to each curve c_{ij} is called an *enhancement* (terminology in accordance with [BCGGM2]) if

$$-\sum_j (\kappa_{ij} + 2) + \sum_{k \in \Sigma_i} w_k^0 \;=\; 4g(\Sigma_i) - 4 \tag{5.1}$$

for each i, where we write $k \in \Sigma_i$, if the k-th decoration point belongs to Σ_i.

Definition 5.1. *A* collapse datum *for* $\mathbf{S}_{\mathbf{w}^0}$ *is a subsurface* Σ *and an enhancement* $\{\kappa_{ij}\}$ *with* $\kappa_{ij} \geq 1$ *for all* (i,j)*. The* collapse *of* Σ *in* $\mathbf{S}_{\mathbf{w}^0}$ *is the weighted DMS* $\overline{\mathbf{S}}_{\mathbf{w}}$ *obtained by filling each boundary* c_{ij} *in* $\mathbf{S}_{\mathbf{w}^0} \setminus \Sigma$ *by a disc with one decorated point that carries the weight* $w_{ij} = \kappa_{ij} - 2$*.*

The condition (5.1) ensures that the weights of $\overline{\mathbf{S}}_{\mathbf{w}}$ indeed satisfy the condition of a wDMS. The case of enhancements $\kappa = 0$ ruled out here is special and requires a different treatment. **For simplicity we consider here only collapse data with all** $\kappa_{ij} \geq 3$**.** (The remaining cases involve mixed angulations with self-folded edges or 2-gons.) A special case of a collapse is a *collision* where the subsurface is topologically a disc. In case of a collision of zeros, the positivity condition for the enhancements in the strong sense, i.e., $\kappa_{ij} \geq 3$, is automatically satisfied.

Consider the special case that $\mathbf{S}_{\mathbf{w}^0} = \mathbf{S}_\Delta$ has simple weights and its subsurface Σ also has simple weights. We can consider Σ as a DMS with κ_{ij} marked points on each boundary component. We denote by $\overline{\mathbf{S}}_{\mathbf{w}}$ the resulting wDMS and thus we can put the three surfaces into a *symbolic short exact sequence*

$$\Sigma \hookrightarrow \mathbf{S}_\Delta \rightsquigarrow \overline{\mathbf{S}}_{\mathbf{w}} . \tag{5.2}$$

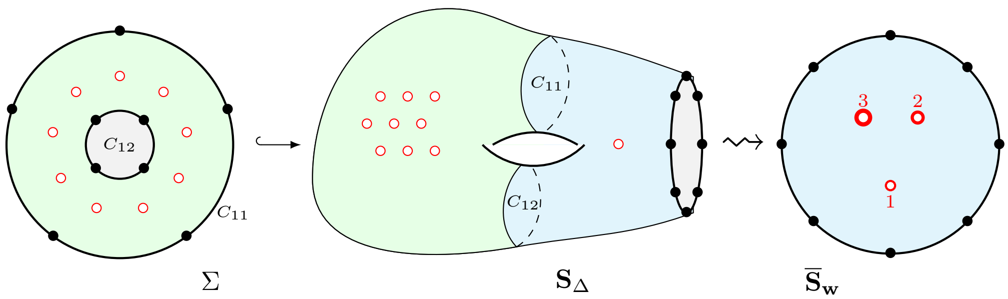

FIGURE 4. A collapse with $\kappa_{11} = 5, \kappa_{12} = 4$.

We formalize the structure induced by a quadratic differential on $\mathbf{S}_{\mathbf{w}}$, generalizing the notion induced in Definition 3.4

Definition 5.2. *A* partial triangulation $\mathbb{A}$ of a collapsed surface $\overline{\mathbf{S}}_{\mathbf{w}}$ *is a collection of open arcs that triangulates the subsurface of* $\mathbf{S}_\Delta$ *whose complement is homeomorphic to* Σ*, and such that each boundary component* c_{ij} *of* Σ *is homotopic in* $\overline{\mathbf{S}}_{\mathbf{w}} \setminus \mathbb{A}$ *to a* $(\kappa_{ij} = w_{ij} + 2)$*-gon, possibly with ends points identified.*

The forward flip *of a partial triangulation* $\mathbb{A}$*, with respect to an arc* $\gamma \in \mathbb{A}$*, is an operation that moves the endpoints of* γ *anti-clockwise (i.e. by a left fractional twist) along two the adjacent sides of the smallest* $\mathbb{A}$*-gon containing* γ*. The inverse of a forward flip is a* backward flip*, which moves the endpoints clockwise.*

Refinements. Let $\mathbb{T} = \{\gamma_j\}_{j \in J}$ be a triangulation of $\mathbf{S}_\Delta$ and $\mathbb{A}$ be a partial triangulation of the collapsed surface $\overline{\mathbf{S}}_{\mathbf{w}}$. We say that $\mathbb{T}$ is a *refinement* of $\mathbb{A}$ if the preimage of $\mathbb{A}$ under $\mathbf{S}_\Delta \rightsquigarrow \overline{\mathbf{S}}_{\mathbf{w}}$ is isotopic to a subset of $\mathbb{T}$. (Note that these preimages are well-defined even though the collapse is not an injective map if a component of Σ has several boundary components.) We let $I = I(\mathbb{T}, \mathbb{A}) \subset J$ be the index set of the *complementary arcs*, the arcs in $\mathbb{T} \setminus \mathbb{A}$.

The same remark justifies:

Definition 5.3. *Let* $(X, q, \psi\colon \mathbf{S}_{\mathbf{w}} \to X^q)$ *be an* $\mathbf{S}_{\mathbf{w}}$*-framed quadratic differential which is saddle-free. Then there is a partial triangulation* $\mathbb{A}_q$ *on* $\overline{\mathbf{S}}_{\mathbf{w}}$ induced from (q, ψ)*, the preimage of the mixed-angulation given by Definition 4.7 under the collapse* $\mathbf{S}_\Delta \rightsquigarrow \overline{\mathbf{S}}_{\mathbf{w}}$*.*

Corollary 5.4. *Two components of* $B_0 \subset \mathrm{FQuad}(\overline{\mathbf{S}}_{\mathbf{w}})$ *can be connected by an arc in* B_2 *with only one point non-saddle-free if and only if the corresponding partial triangulations are related by a forward flip.*

Proof. Take the preimage of the construction in Proposition 4.8 under the collapse map. $\square$

5.2. **Exchange graphs and spherical twists.** The mutation (of quivers), tilting (of categories) and flipping (of edges) operations give rise to a number of exchange graphs that we summarize here.

- The *unoriented exchange graph* $\underline{\mathrm{EG}}(\mathbf{S})$ has vertices corresponding to triangulations of $\mathbf{S}$ and edges corresponding to *flips*.
- Given a mutation equivalence class $\mathfrak{Q}$ of a quiver, the *unoriented cluster exchange graph* $\underline{\mathrm{CEG}}(\mathfrak{Q})$ is the oriented graph whose vertices are cluster tilting objects in $\mathcal{C}(\mathfrak{Q})$ and whose edges are mutations between them (see [Kel11a] for more details).

For the second definition note that mutation equivalences above identify all the associated cluster categories (without nontrivial autoequivalences as monodromy). Hence the symbol $\mathcal{C}(\mathfrak{Q})$ is well-defined. In general underlined symbols correspond to unoriented (exchange) graphs. We need the oriented version of these graphs:

- The *exchange graph* $\mathrm{EG}(\mathbf{S})$ *of (an undecorated) surface* $\mathbf{S}$ is obtained from $\underline{\mathrm{EG}}(\mathbf{S})$ by replacing each unoriented edge with a 2-cycle.
- Similarly, the oriented version $\mathrm{CEG}(\mathfrak{Q})$ is obtained from $\underline{\mathrm{CEG}}(\mathfrak{Q})$ by replacing each unoriented edge with a 2-cycle.
- The *exchange graph of the wDMS* $\mathbf{S}_\mathbf{w}$ is the directed graph $\mathrm{EG}(\mathbf{S}_\mathbf{w})$ whose vertices are partial triangulations and whose oriented edges are forward flips between them.
- The *(total) exchange graph* $\mathrm{EG}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ is the oriented graph whose vertices are all hearts in $\mathcal{D}$ and whose directed edges correspond to simple forward tiltings between them (Section 2.1). We abbreviate $\mathrm{EG}(\Gamma) \coloneqq \mathrm{EG}(\mathrm{pvd}(\Gamma))$.

We usually focus attention on *a connected component* $\mathrm{EG}^\circ(\Gamma)$ of the exchange graph $\mathrm{EG}(\mathrm{pvd}(\Gamma))$, called the *principal component*, consisting of those hearts that are reachable by repeated simple tilting from the canonical heart $\mathcal{H}(\Gamma)$ in the quiver case for $\Gamma = \Gamma(Q, W)$. Similarly, we write $\mathrm{EG}^\circ(\mathbf{S}_\mathbf{w})$ for a connected component of the surface exchange graph. We also write $\mathrm{EG}^\circ(\overline{\mathbf{S}}_\mathbf{w})$ to indicate that the wDMS is obtained by a subsurface collapse.

Recall that a graph is called (m_1, m_2)*-regular*, if each vertex has m_1 outgoing edges and m_2 incoming edges. By definition the graphs $\mathrm{EG}(\mathbf{S}_\mathbf{w})$ and $\mathrm{EG}(\mathcal{D})$ are (m, m)-regular with m being the number of arcs of the mixed-angulation or of the partial triangulation or the rank of $K(\mathcal{D})$ respectively.

We start the comparison of these graphs in the coarse (undecorated) cases. If two triangulations are related by a flip, then both the corresponding quivers with potential are related by a mutation, in the sense of [FST08; DWZ08].

Theorem 5.5 ([FST08])**.** *There is an isomorphism* $\underline{\mathrm{EG}}(\mathbf{S}) \cong \underline{\mathrm{CEG}}(\mathbf{S})$ *of the unoriented (triangulation) exchange graphs and cluster exchange graphs. This isomorphism upgrades to an isomorphism* $\mathrm{EG}(\mathbf{S}) \cong \mathrm{CEG}(\mathbf{S})$*.*

Spherical twist groups. For further graph comparison we let $\mathrm{ST}(\Gamma) \leq \mathrm{Aut}(\mathrm{pvd}(\Gamma))$ be the *spherical twist group* of $\mathrm{pvd}(\Gamma)$, that is the subgroup generated by the set of twists $\{\Phi_S \mid S \in \mathrm{Sim}\,\mathcal{H}(\Gamma)\}$, where the *twist functor* Φ_S is defined by

$$\Phi_S(X) = \mathrm{Cone}\,(S \otimes \mathrm{Hom}^\bullet(S, X) \to X) \tag{5.3}$$

Note that $\mathrm{ST}(\Gamma)$ is in fact generated by spherical twists along all reachable spherical objects, that is all simples in some $\mathcal{H} \in \mathrm{EG}^\circ\,\mathrm{pvd}(\Gamma)$, see [Qiu16, § 2.2].

For a heart $\mathcal{H} \in \mathrm{EG}^\circ(\mathrm{pvd}(\Gamma))$ we denote by $\mathrm{EG}^\circ[\mathcal{H}, \mathcal{H}[1]]$ the full subgraph whose vertices are intermediate hearts $\mathcal{H} \le \mathcal{H}' \le \mathcal{H}[1]$. The following result is [KQ20, Theorem 2.10], based on the unpublished result of Keller-Nicolás announced in [Kel11a, Theoreom 5.6].

Theorem 5.6. *Let Γ be the Ginzburg dg algebra of some non-degenerate quiver with potential (Q, W). There is a covering of oriented graphs*

$$\mathrm{EG}^\circ(\mathrm{pvd}(\Gamma))/\,\mathrm{ST}(\Gamma) \cong \mathrm{CEG}(\Gamma). \tag{5.4}$$

The fundamental domain of $\mathrm{EG}^\circ(\mathrm{pvd}(\Gamma))/\,\mathrm{ST}(\Gamma)$ is $\mathrm{EG}^\circ[\mathcal{H}, \mathcal{H}[1]]$ for any heart $\mathcal{H} \in \mathrm{EG}^\circ(\mathrm{pvd}(\Gamma))$, in the sense that there is an isomorphism between unoriented graph

$$\underline{\mathrm{EG}^\circ}[\mathcal{H}, \mathcal{H}[1]] \cong \underline{\mathrm{CEG}}(\Gamma),$$

where $\underline{\mathrm{EG}^\circ}$ denotes the underlying unoriented graph of EG°.

5.3. **Braid groups.** Two types of braid groups provide the relation between the various exchange graphs appearing here.

Surface braid groups. One of the standard definitions of the surface braid group $\mathrm{SBr}(\mathbf{S}_\Delta)$ of a DMS (with non-empty boundary) is as the fundamental group of the configuration space $\mathrm{conf}_\Delta(\mathbf{S})$ of $|\Delta|$ (unordered) points in $\mathbf{S}$. It is a well-known theorem (see e.g. [GJP15, Section 2.4, equation (5)]) that the surface braid group is a subgroup of mapping class groups

$$\mathrm{SBr}(\mathbf{S}_\Delta) := \pi_1 \,\mathrm{conf}_\Delta(\mathbf{S}) = \ker\left(\mathrm{MCG}(\mathbf{S}_\Delta) \xrightarrow{F_*} \mathrm{MCG}(\mathbf{S})\right), \tag{5.5}$$

where F_* is induced by the forgetful map $F\colon \mathbf{S}_\Delta \to \mathbf{S}$, forgetting the decorations. There is a natural isomorphism between graphs

$$\mathrm{EG}(\mathbf{S}_\Delta)/\,\mathrm{SBr}(\mathbf{S}_\Delta) = \mathrm{EG}(\mathbf{S}), \tag{5.6}$$

induced by the induced map $F : \mathrm{EG}(\mathbf{S}_\Delta) \to \mathrm{EG}(\mathbf{S})$.

While $\mathrm{SBr}(\mathbf{S}_\Delta)$ is the traditional generalization of the classical braid group, we need a (normal) subgroup of it, since we would like to restrict $\mathrm{EG}(\mathbf{S}_\Delta)$ in (5.6) to a connected component.

Braid twist groups. For any closed arc $\eta \in \mathrm{CA}(\mathbf{S}_\Delta)$, there is a (positive) *braid twist* $\mathrm{B}_\eta \in \mathrm{MCG}(\mathbf{S}_\Delta)$ along η, as shown in Figure 5. The *braid twist group* $\mathrm{BT}(\mathbf{S}_\Delta)$ of the decorated marked surface $\mathbf{S}_\Delta$ is the subgroup of $\mathrm{MCG}(\mathbf{S}_\Delta)$ generated by the braid twists B_η for all $\eta \in \mathrm{CA}(\mathbf{S}_\Delta)$.

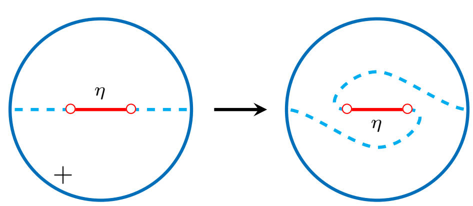

FIGURE 5. The braid twist B_η

Let $\mathbb{T}$ be a triangulation of the decorated surface $\mathbf{S}_\Delta$ consisting of open arcs. The *dual graph* $\mathbb{T}^*$ of $\mathbb{T}$ is then a collection of closed arcs η. By [Qiu16, Lemma 4.2], $\{\mathrm{B}_\eta \mid \eta \in \mathbb{T}^*\}$ is a set of generators of $\mathrm{BT}(\mathbf{S}_\Delta)$.

For later use, we give a characterization of $\mathrm{BT}(\mathbf{S}_\Delta)$. By (5.5), any $\xi \in \mathrm{SBr}(\mathbf{S}_\Delta)$ corresponds to $|\Delta|$ paths p_i on $\mathbf{S}_\Delta$. Their union $\coprod p_i$ forms a collection of cycles in $\mathbf{S}$. The product of these cycles gives a well-defined element in $\mathrm{H}_1(\mathbf{S})$ (more details are given in the forthcoming paper [Qiu24]) and we obtain a map, called *topological Abel-Jacobi map* $\mathrm{AJ} = \mathrm{AJ}_{\mathbf{S}_\Delta} \colon \mathrm{SBr}(\mathbf{S}_\Delta) \to \mathrm{H}_1(\mathbf{S})$.

Lemma 5.7. *The braid twist group* $\mathrm{BT}(\mathbf{S}_\Delta)$ *is precisely the kernel of the map* AJ.

Proof. By [QZ20, Proposition 2.7, in particular Figure 4], the group $\mathrm{SBr}(\mathbf{S}_\Delta)$ admits a set of generators σ_i for $1 \le i \le |\Delta| - 1$, and δ_r for $1 \le r \le 2g + b - 1$. Here the σ_i are are braid twists along a collection of arcs connecting the marked points (within a topological discs). The δ_r are point-pushing diffeomorphisms around simple closed curves based at the first marked point, namely the $2g$ curves of a canonical dissection and $b-1$ curves around the boundary components. We denote by H the subgroup of $\mathrm{SBr}(\mathbf{S}_\Delta)$ generated by δ_r, which is isomorphic to $\pi_1(\mathbf{S}, Z_1)$. In particular H is a free group and $H/[H,H] \cong \mathrm{H}_1(\mathbf{S})$. By definition, $\mathrm{BT}(\mathbf{S}_\Delta)$ is a normal subgroup of $\mathrm{SBr}(\mathbf{S}_\Delta)$ and contained in $\ker \mathrm{AJ}$. Thus we express the map AJ as

$$\mathrm{SBr}(\mathbf{S}_\Delta) \;=\; \mathrm{BT}(\mathbf{S}_\Delta) \cdot H \to H/[H,H],$$

sending the generators of $\mathrm{BT}(\mathbf{S}_\Delta)$ to the neutral element and the elements in H to their classes modulo $[H,H]$. This implies that $\mathrm{BT}(\mathbf{S}_\Delta) \cap H \le [H,H]$. A direct calculation shows that $[\delta_s, \delta_r]$ is in $\mathrm{BT}(\mathbf{S}_\Delta)$ for any $1 \le s < r \le 2g+b-1$. In fact, if we change generators for convenience as in [QZ20, Proposition 3.1, in particular Figure 7] and define (setting $\varepsilon_0 = 1$)

$$\varepsilon_r \;=\; \begin{cases} \delta_r \varepsilon_{r-1} & \text{if } r \notin 2\mathbb{N}_{\le g} \\ \delta_r \varepsilon_{r-2} & \text{if } r \in 2\mathbb{N}_{\le g}, \end{cases} \quad \text{i.e.} \quad \delta_r \;=\; \begin{cases} \varepsilon_r \varepsilon_{r-1}^{-1} & \text{if } r \notin 2\mathbb{N}_{\le g} \\ \varepsilon_r \varepsilon_{r-2}^{-1} & \text{if } r \in 2\mathbb{N}_{\le g}, \end{cases}$$

then such a commutator equals (using $\tau_j = \epsilon_j \sigma_1 \epsilon_j^{-1}$, as drawn in [QZ20, Figure 8])

$$[\epsilon_s, \epsilon_r] \;=\; \begin{cases} (\tau_s b)^{-1} a (\tau_r a \tau_s)^{-1} a b \tau_r b & \text{if } s+1 \in 2\mathbb{N}_{\le g}, \\ (b b \tau_s b)^{-1} a \tau_r a^{-1} \tau_s a b \tau_r b & \text{otherwise}, \end{cases}$$

where $a = \sigma_2 \sigma_1 \sigma_2^{-1}$ and $b = \sigma_2$. Here the cases depend on the relative position of τ_s and τ_r at Z_2. (More precisely, if $s < r$ the first case occurs precisely if τ_r is before τ_s in the counterclockwise order of a neighborhood of Z_2 slit along the arc σ_1.) This implies that $[H,H] \le \mathrm{BT}(\mathbf{S}_\Delta)$ and hence that $\mathrm{BT}(\mathbf{S}_\Delta) \cap H = [H,H]$ or equivalently $\mathrm{BT}(\mathbf{S}_\Delta) = \ker \mathrm{AJ}$, as claimed. □

We can now summarize the whole discussion in the following two theorems. The first restricts (5.6) to a connected component.

Theorem 5.8. *There is an isomorphism* $\mathrm{EG}^\circ(\mathbf{S}_\Delta)/\mathrm{BT}(\mathbf{S}_\Delta) = \mathrm{EG}(\mathbf{S})$ *between the exchange graph of the undecorated surface and the braid twist orbits of the exchange graph of the decorated surface.*

Proof. This is the content of [Qiu16, Remark 3.10]. In fact, Lemma 3.9 in loc. cit. shows that there is a well-defined surjective map $\mathrm{EG}^\circ(\mathbf{S}_\Delta)/\mathrm{BT}(\mathbf{S}_\Delta) \to \mathrm{EG}(\mathbf{S})$. To show injectivity it suffices to know that the directed graph of intermediate hearts is a fundamental domain for the $\mathrm{BT}(\mathbf{S}_\Delta)$-action. The Lemma 3.8 in loc. cit. shows that composition of two forward flips is a braid twist and completes the proof. For the claim on fundamental domains we apply [KQ15, Proposition 8.3]. (We can't apply Theorem 5.6 since the current theorem is used in its proof.) □

The twist groups in the preceding theorems can be identified and the corresponding isomorphism can be lifted.

Theorem 5.9. *[Qiu16; Qiu18] There is an isomorphism* $\mathrm{ST}(\Gamma_{\mathbb{T}}) \cong \mathrm{BT}(\mathbb{T})$ *between the twist groups, sending the standard generators to the standard generators. Thus the isomorphism (between oriented graphs) in Theorem 5.5 lifts to an isomorphism*

$$\mathrm{EG}^\circ \operatorname{pvd}(\Gamma_{\mathbb{T}}) \;\cong\; \mathrm{EG}^\circ(\mathbf{S}_\Delta)\,. \tag{5.7}$$

As a consequence, we have $\mathrm{EG}^\circ \operatorname{pvd}(\Gamma_{\mathbb{T}})/\operatorname{ST}(\mathbf{S}_\Delta) \cong \mathrm{EG}(\mathbf{S})$.

5.4. **Principal parts of exchange graphs.** In order to use the preceding results on $\mathrm{EG}(\mathbf{S}_\Delta)$ to explore the graph $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$ we need to relate partial triangulations and triangulations.

Principal parts. Let us fix an initial triangulation $\mathbb{T}_0$ of $\mathbf{S}_\Delta$ and let $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$ be the principal connected component of $\mathrm{EG}(\mathbf{S}_\Delta)$ containing $\mathbb{T}_0$. We define the *principal part* $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ of $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$ to be the full subgraph of $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$ consisting of the partial triangulations which admit a refinement that belongs to the component $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$. Note that

- we do not claim that $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ is connected. Moreover,
- a priori it is not even clear if $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ consists of connected components. That is, it is not a priori clear that vertices in $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ that are connected through $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$ are in fact connected through $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$.

Connectedness of refinements. Next, we show that when restricted to principal part, certain connectedness property holds.

Proposition 5.10. *Let* $\mathbb{A}$ *be a partial triangulation in* $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$. *The full subgraph of the exchange graph* $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$ *consisting of refinements of* $\mathbb{A}$ *is connected.*

Proof. Without loss of generality we only need to consider the case when Σ has one connected component. Take any two refinements $\mathbb{T}_1$ and $\mathbb{T}_2$ of $\mathbb{A}$ in $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$. Let T_1, T_2 be their images in $\mathrm{EG}(\mathbf{S})$ under the forgetful map $F\colon \mathbf{S}_\Delta \to \mathbf{S}$. By [Hat91], there is a flip sequence connecting the triangulation T_2 and T_1 in the complement of $F(\mathbb{A})$. Such a sequence lifts to a flip sequence of refinements of $\mathbb{A}$ from $\mathbb{T}_2$ to some triangulation $\mathbb{T}_1'$ with the property that $F(\mathbb{T}_1') = T_1 = F(\mathbb{T}_1)$. Then $\mathbb{T}_1$ and $\mathbb{T}_1'$ differ by an element b of $\mathrm{BT}(\mathbf{S}_\Delta)$ by Theorem 5.8 since these triangulations are both in the principal component $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$. Moreover, b preserves $\mathbf{S}_\Delta \setminus \Sigma$ pointwise as $\mathbb{T}_1$ and $\mathbb{T}_1'$ are both refinements of $\mathbb{A}$. By Lemma 5.11, we know that b is actually in $\mathrm{BT}(\Sigma)$. By Theorem 5.8 again, the two triangulations of Σ induced by $\mathbb{T}_1$ and $\mathbb{T}_1'$ are connected by a flip sequence that lifts to a flip sequence from $\mathbb{T}_1$ to $\mathbb{T}_1'$ in the refinements of $\mathbb{A}$. Composing the two flip sequences implies the claim. □

Lemma 5.11. *If an element* b *in* $\mathrm{BT}(\mathbf{S}_\Delta)$ *preserves* $\mathbf{S}_\Delta \setminus \Sigma$ *pointwise, then* b *is actually in* $\mathrm{BT}(\Sigma)$.

Proof. Since the element b preserves $\mathbf{S}_\Delta \setminus \Sigma$ pointwise, it belongs to $\mathrm{SBr}(\Sigma)$. We conclude that

$$b \in \mathrm{BT}(\mathbf{S}_\Delta) \cap \mathrm{SBr}(\Sigma) \;=\; \ker \mathrm{AJ}_{\mathbf{S}_\Delta}|_{\mathrm{SBr}(\Sigma)} = \;\ker \mathrm{AJ}_\Sigma \;=\; \mathrm{BT}(\Sigma)$$

by Lemma 5.7. □

Remark 5.12. *In the case of a collision, i.e., when Σ is a disk (or the disjoint union of many disks), the exchange graph $\mathrm{EG}(\Sigma)$ is already connected. Then the lemma above holds automatically.*

We end this section with a proposition showing that there exists a refinement of a flip of a partial triangulation in an appropriate sense.

Proposition 5.13. *Let $\mathbb{A}$ be a partial triangulation in $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$. Any forward flip $\mathbb{A} \xrightarrow{\gamma} \mathbb{A}^\sharp_\gamma$ in $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$ can be refined to a forward flip $\mathbb{T} \xrightarrow{\gamma} \mathbb{T}^\sharp_\gamma$ in $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$. That is, $\mathbb{A}$ can be refined to a triangulation $\mathbb{T}$ such that the γ-forward flip of $\mathbb{T}$ composed with forgetting the complementary arcs is the same as the γ-forward flip in $\mathbb{A}$.*

The same statement holds, with 'forward flip' replaced throughout by 'backward flip'. In particular the principal part $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ is a union of connected components of $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$.

Yet another restatement of the first statement of the proposition is that any forward flip of an arc γ in a partial triangulation $\mathbb{A}$ in $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ leads again to a partial triangulation in $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$.

Proof. Let $\gamma^\sharp$ be the new arc in $\mathbb{A}^\sharp_\gamma$. The vertices at the end points of γ and $\gamma^\sharp$ form a quadrilateral Q in $\overline{\mathbf{S}}_\mathbf{w}$. Two of its edges are the counterclockwise adjacent edges of γ in the $\mathbb{A}$-polygon P_0 in $\overline{\mathbf{S}}_\mathbf{w}$ containing γ. These two adjacent edges and γ forms two angles a and b, drawn in red in Figure 6. The other two edges are not necessarily in $\mathbb{A}^\sharp_\gamma$, see the green dashed arcs in Figure 6. We only need to refine $\mathbb{A}$

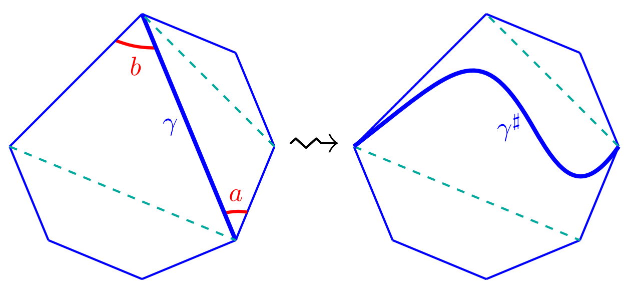

FIGURE 6. Refinement of a flip $\mathbb{A} \xrightarrow{\gamma} \mathbb{A}^\sharp_\gamma$ (collision case)

to a triangulation $\mathbb{T}$ of $\mathbf{S}_\Delta$ so that the angles a and b are not cut by the new added arcs. If it is a collision, Figure 6 shows that by including the green dashed arcs mentioned above in the refinement, the job is done.

In general, the decoration in the $\mathbb{A}$-polygon P_0 containing the angle a is obtained from a boundary component ∂_0 of Σ, cf. Figure 7. Then by identifying the marked points M_i in this component ∂_0 with the vertices of P_0, the angle a corresponds to some angle a between segments of ∂_0. As $\kappa_{ij} \geq 3$ one can refine $\mathbb{A}$ (by choosing a triangulation of Σ) so that the angle a (and similarly for b) is not cut by new added arcs as required. (In fact $\kappa_{ij} \geq 2$ is enough, but for $\kappa_{ij} = 1$ the green arc in Figure 7 might not exist.)

The backward flip statement is proven the same way using the quadrilateral formed by the end points of γ and $\gamma^\flat$. As a consequence, the graph $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ is (m, m)-regular, where m is the number of arcs in any mixed-angulation of $\overline{\mathbf{S}}_\mathbf{w}$. Since we already remarked the same statement for $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$, the second claim of the proposition follows. □

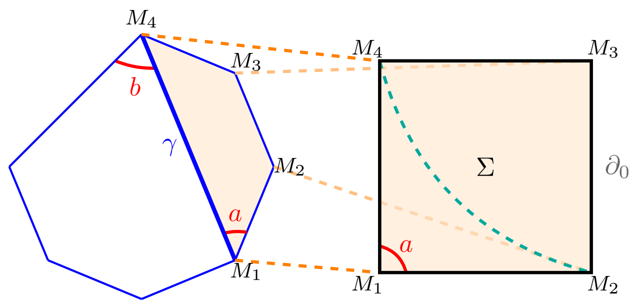

FIGURE 7. Refinement of a flip $\mathbb{A} \xrightarrow{\gamma} \mathbb{A}^\sharp_\gamma$ (in general)

5.5. **Example of non-connectedness.** We finish this section by giving an example of exchange graphs, showing that $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ is not connected in general. We define $\mathrm{EG}_\mathbf{w}(\overline{\mathbf{S}})$ to be the exchange graph of partial triangulations of the undecorated collapsed surface $\overline{\mathbf{S}}$. This graph is easy to draw if the mapping class group of $\overline{\mathbf{S}}$ is finite and captures some connectivity information of the principal part for the following reason:

Lemma 5.14. *The forgetful map $F\colon \mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})/\,\mathrm{SBr}(\overline{\mathbf{S}}_\mathbf{w}) \to \mathrm{EG}_\mathbf{w}(\overline{\mathbf{S}})$ is surjective and hence an isomorphism. As a result, if $\mathrm{EG}_\mathbf{w}(\overline{\mathbf{S}})$ is not connected, neither is $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$.*

Proof. Given any partial triangulation $\underline{\mathbb{A}}$ in $\mathrm{EG}_\mathbf{w}(\overline{\mathbf{S}})$, one can refine it to a triangulations T of $\mathbf{S}$. By [Hat91], the exchange graph $\mathrm{EG}(\mathbf{S})$ of the undecorated (non-collapsed) surface with simple weights $\mathbf{S}$ is connected and thus $\mathrm{T} \in \mathrm{EG}(\mathbf{S})$ lifts to a triangulation $\mathbb{T}$ in the principal component $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$ with $F(\mathbb{T}) = \mathrm{T}$. Restricting $\mathbb{T}$ back to $\overline{\mathbf{S}}_\mathbf{w}$, we obtain a partial triangulation $\mathbb{A} \in \mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ with $F(\mathbb{A}) = \underline{\mathbb{A}}$. $\square$

Example 5.15. *Let $\overline{\mathbf{S}}_\mathbf{w}$ be a torus with one boundary component ∂ and one decoration with weight $\mathbf{w} = 3$. Then $\mathrm{EG}_\mathbf{w}(\overline{\mathbf{S}})$ and hence $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ are not connected.*

Proof. Let $\overline{\mathbf{S}}$ be the undecorated torus with boundary circle ∂. We identify a fundamental domain of the universal cover $\overline{\mathbf{S}}$ with the unit square in $\mathbb{R}^2$ with ∂ being a (real) bubble at the corner of first quadrant. The first homology of this surface is simply $H_1(\overline{\mathbf{S}}) = \mathbb{Z}^2$. We denote by $D_{p,q}$ the Dehn twist along an oriented simple closed curve $C_{p,q}$ with homology class $H_1(C_{p,q}) = (p,q)$ for $(p,q) \in \mathbb{Z}^2$ satisfying $\gcd(p,q) = 1$. The mapping class group of $\overline{\mathbf{S}}$ is the group

$$\mathrm{MCG}(\overline{\mathbf{S}}) \;=\; \langle X, Y\rangle/(XYX - YXY) \;\cong\; \mathrm{Br}_3\,,$$

generated by $X = D_{1,0}$ and $Y = D_{0,1}$. Note that the Dehn twist $D_\partial := (XY)^6$ is in the center of $\mathrm{MCG}(\overline{\mathbf{S}})$.

A partial triangulation $\underline{\mathbb{A}}$ of $\mathrm{EG}_\mathbf{w}(\overline{\mathbf{S}})$ in this case is just a $\mathbf{w}$-mixed-angulation, a pentagon with edges γ_h, γ_v, ∂, such that glueing edges different from ∂ yields a torus. We oriented them so that $\overrightarrow{\gamma_h}, \overrightarrow{\gamma_v}, -\overrightarrow{\gamma_h}, -\overrightarrow{\gamma_v}$ are in anticlockwise order, cf. Figure 8.

To show non-connectivity of $\mathrm{EG}_\mathbf{w}(\overline{\mathbf{S}})$ we coarsify the datum given by a mixed-angulation and find an invariant. First, up to composition with an element in the normal subgroup generated by D_∂, a mixed-angulation $\underline{\mathbb{A}}$ is determined by a 2-by-2

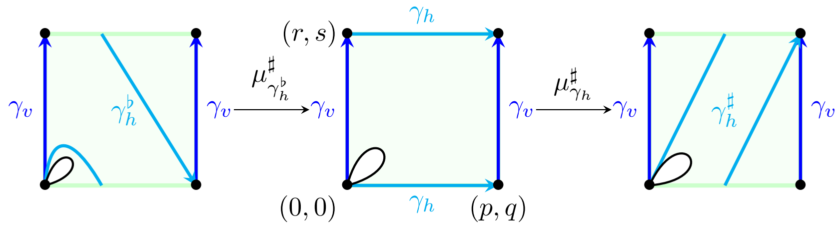

FIGURE 8. The forward flips of the pentagon on torus

matrix with rows $\overrightarrow{h} = H_1(\overrightarrow{\gamma_h})$ and $\overrightarrow{v} = H_1(\overrightarrow{\gamma_v})$, together with the location (at which of the four corners) of the boundary ∂. We represent the position by boxing the corresponding element in the matrix, called the *bubble*. With our orientation conventions for the arcs, each mixed-angulation is coarsely represented by one of the following four matrices:

$$\begin{pmatrix} p & q \\ \boxed{r} & s \end{pmatrix} \cong \begin{pmatrix} \boxed{r} & s \\ -p & -q \end{pmatrix} \cong \begin{pmatrix} -p & \boxed{-q} \\ -r & -s \end{pmatrix} \cong \begin{pmatrix} -r & -s \\ p & \boxed{q} \end{pmatrix}. \tag{5.8}$$

We shall use, as the normal form, that the bubble is in the left bottom corner. In such a form the matrix is $\begin{pmatrix} \overrightarrow{h} \\ \overrightarrow{v} \end{pmatrix} = \begin{pmatrix} p & q \\ \boxed{r} & s \end{pmatrix}$. Second, we coarsify by considering all matrix entries modulo $3\mathbb{Z}$, and we show that $\overrightarrow{h} + \overrightarrow{v} \in (\mathbb{Z}_3)^2$ is constant on each connected component.

The flips in Figure 8 can be represented as

$$\begin{pmatrix} \boxed{p-r} & q-s \\ r & s \end{pmatrix} \xrightarrow[M_X\cdot]{\mu^\sharp_{\gamma_h^\flat}} \begin{pmatrix} p & q \\ \boxed{r} & s \end{pmatrix} \xrightarrow[M_X^2\cdot]{\mu^\sharp_{\gamma_h}} \begin{pmatrix} \boxed{p+2r} & q+2s \\ r & s \end{pmatrix},$$

where, on the level of matrixes, the two flips (i.e., the forward flip at $\gamma_h^\flat$, resp. at γ_h) to, resp. starting at, $\underline{\mathbb{A}}$ are represented by multiplying on the left by $M_X = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and M_X^2 respectively. Changing to the normal form, we have

$$\begin{pmatrix} -r & -s \\ \boxed{p-r} & q-s \end{pmatrix} \xrightarrow[M_X'\cdot]{\mu^\sharp_{\gamma_h^\flat}} \begin{pmatrix} p & q \\ \boxed{r} & s \end{pmatrix} \xrightarrow[M_X''\cdot]{\mu^\sharp_{\gamma_h}} \begin{pmatrix} -r & -s \\ \boxed{p+2r} & q+2s \end{pmatrix},$$

for $M_X' = \left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $M_X'' = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 2 \end{smallmatrix}\right)$. It is now straightforward to check that $\overrightarrow{h} + \overrightarrow{v} \in (\mathbb{Z}_3)^2$ remains unchanged. Similarly, the other two flips to/at $\underline{\mathbb{A}}$ can be represented as

$$\begin{pmatrix} r-p & s-q \\ \boxed{-p} & -q \end{pmatrix} \xrightarrow[M_Y'\cdot]{\mu^\sharp_{\gamma_v^\flat}} \begin{pmatrix} p & q \\ \boxed{r} & s \end{pmatrix} \xrightarrow[M_Y''\cdot]{\mu^\sharp_{\gamma_v}} \begin{pmatrix} r+2p & s+2q \\ \boxed{-p} & -q \end{pmatrix}$$

for $M_Y' = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ and $M_Y'' = \left(\begin{smallmatrix} 2 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The row $\overrightarrow{h} + \overrightarrow{v} \in (\mathbb{Z}_3)^2$ is preserved again.

Using $\overrightarrow{h} + \overrightarrow{v} \in (\mathbb{Z}_3)^2$ as an invariant for the selected connected component of the exchange graph, we see that the mixed-angulations $\begin{pmatrix} 1 & 0 \\ \boxed{0} & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ \boxed{0} & 1 \end{pmatrix}$ are not in the same connected component of $\mathrm{EG}_{\mathbf{w}}(\overline{\mathbf{S}})$. $\square$

6. Categorification of collapses

In this section, we categorify the constructions in Section 5, by associating with a collapse a quotient category. For computations it is convenient to express this quotient category in terms of triangulations. The main point of this section is to analyse a subset of hearts of bounded t-structures of the quotient category that we call of quotient type with respect to the subcategory that has been collapsed. The leads to a notion of exchange graphs of these quotient type hearts. The goal of this section, Theorem 6.9, is to show that the principal part of this exchange graph agrees with the principal part of the exchange of partial triangulations we introduced previously.

6.1. **The quotient categories associated to collapsed surfaces.** We have been associating in Section 3.3 a CY_3-category $\mathrm{pvd}(\Gamma_{\mathbb{T}})$ to a triangulation $\mathbb{T}$ of a wDMS $\mathbf{S}_\Delta$ with simple weights. Theorem A1 in [BQZ21] shows that this category $\mathrm{pvd}(\Gamma_{\mathbb{T}})$ is in fact canonically associated with $\mathbf{S}_\Delta$, i.e. the derived equivalences given by Proposition 2.7 can be identified consistently. We call it $\mathcal{D}_3(\mathbf{S}_\Delta)$. Thus the inclusion $\Sigma \subset \mathbf{S}_\Delta$ in (5.2), together with the discussion in 2.3 and 3.3, induces a short exact sequence of triangulated categories:

$$0 \longrightarrow \mathcal{D}_3(\Sigma) \longrightarrow \mathcal{D}_3(\mathbf{S}_\Delta) \longrightarrow \mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}) \longrightarrow 0. \tag{6.1}$$

Equivalently, we define the category $\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})$ as the Verdier quotient $\mathcal{D}_3(\mathbf{S}_\Delta)/\mathcal{D}_3(\Sigma)$. We will now give a more concrete construction of $\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})$ by choosing (partial) triangulations and show that it is indeed independent of the choices.

Triangulation of subsurfaces. If $\mathbb{T}$ is any triangulation of $\mathbf{S}_\Delta$, we can homotope the arcs to pass each through one of the marked points on $\partial\Sigma$. In this way, the subsurface inherits a triangulation $\mathbb{T}|_\Sigma$. This triangulation is obviously a refinement of the mixed-angulation $\mathbb{A}$ obtained by forgetting the edges in $\mathbb{T}|_\Sigma$ and collapsing to $\overline{\mathbf{S}}_{\mathbf{w}}$.

This defines an inclusion of triangulated categories $\mathrm{pvd}(\mathbb{T}|_\Sigma) \to \mathrm{pvd}(\mathbb{T})$. Any other refinement of $\mathbb{A}$ differs from $\mathbb{T}$ by a sequence of flips, see Proposition 5.10, i.e. of mutations in the vertices of the corresponding subquiver. Then the quotient category $\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})$ can be realized as $\mathrm{pvd}(\mathbb{T})/\,\mathrm{pvd}(\mathbb{T}|_\Sigma)$. The independence of $\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})$ of the chosen refinement is given by the following proposition.

Proposition 6.1. *Let $I \subseteq Q_0 = \{1, \dots, n\}$ be a non-empty subset, and μ be a sequence of mutations at vertices $k_j \in I$ (possibly repeated), then we have the following equivalence*

$$\mathrm{pvd}(Q, W)/\,\mathrm{pvd}\big((Q, W)_I\big) \simeq \mathrm{pvd}\big(\mu(Q, W)\big)/\,\mathrm{pvd}\big((\mu(Q, W))_I\big)$$

of quotient triangulated categories.

Proof. Let $I^{(0)} = I \subset Q_0 = \{1, \dots, n\}$ be a non-empty subset. For simplicity we omit the potentials in the proof. Denote the quiver Q as $Q^{(0)}$. For $j \geq 0$, let $Q^{(j+1)} = \mu_{k_j} Q^{(j)}$, for some $k_j \in Q_0^{(j)}$. We know that $(Q^{(j)}, W^{(j)})$ are all right-equivalent and their associated triangulated categories $\mathrm{pvd}(Q^{(j)})$ are equivalent, see [KY11]. The equivalence also holds for $\mathrm{pvd}(Q^{(j)}{}_{|I})$ and $\mathrm{pvd}(\mu_{k_j}(Q^{(j)}{}_{|I}))$. By the mutation-restriction compatibility, the latter is the same as $\mathrm{pvd}((\mu_{k_j} Q^{(j)})_{|I})$

and the equivalence in the bottom level of the following diagram is compatible with the one above

$$\begin{tikzcd}
\mathrm{pvd}(Q^{(j)}) \arrow[rr, no head, "\simeq", "\text{[KY11]}"'] & & \mathrm{pvd}\left(Q^{(j+1)}\right) \\
\bigcup & & \bigcup \\
\mathrm{pvd}\left(Q^{(j)}{}_{|I}\right) \arrow[r, no head, "\simeq", "\text{[KY11]}"'] & \mathrm{pvd}\left(\mu_{k_j}(Q^{(j)}{}_{|I})\right) \arrow[r, no head, "\simeq", "\text{[LF09]}"'] & \mathrm{pvd}\left(Q^{(j+1)}{}_{|I}\right)
\end{tikzcd}$$

The diagram is therefore commutative. For the statement to follow, is then enough to know that the Ginzburg category associated to a sub-quiver is a thick triangulated sub-category of the Ginzburg category attached to the whole quiver, hence the quotient is well-defined. □

6.2. **Partial triangulations induce hearts of quotient type.** Now we focus on hearts of bounded t-structures in the quotient category $\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})$. We will restrict our attention to certain hearts that we call of quotient type. We show that partial triangulation induce hearts of quotient type through the choice of a refinement. We start with a general fact.

Proposition 6.2 ([AGH19, Proposition 2.20])**.** *Let $i : \mathcal{C} \to \mathcal{D}$ be a t-exact fully faithful functor of triangulated categories equipped with bounded t-structures, with a well-defined quotient functor $j : \mathcal{D} \to \mathcal{D}/\mathcal{C}$. Let $\mathcal{H}_{\mathcal{D}}$ and $\mathcal{H}_{\mathcal{C}}$ be the two hearts in $\mathcal{D}$ and $\mathcal{D}$ respectively. Then the following are equivalent*

a) the essential image $i(\mathcal{H}_{\mathcal{C}}) \subset \mathcal{H}_{\mathcal{D}}$ is a Serre subcategory, and
b) the quotient $\mathcal{D}/\mathcal{C}$ has a bounded t-structure such that j is t-exact, whose heart is equivalent to $\mathcal{H}_{\mathcal{D}}/\mathcal{H}_{\mathcal{C}}$.

The (bounded) t-structure corresponding to $\mathcal{H}_{\mathcal{D}}/\mathcal{H}_{\mathcal{C}}$ in $\mathcal{D}/\mathcal{C}$ of point b) is described in [AGH19, Proposition 2.20].

Definition 6.3. *If a heart on a quotient triangulated category arises as described by [AGH19, Proposition 2.20], we say that it is of* quotient type*. We say moreover that it is* induced *by the hearts in $\mathcal{D}$ and $\mathcal{C}$, or* induced by the heart on $\mathcal{D}$*, if that heart on $\mathcal{C}$ is obtained by restriction.*

Note that, a priori, a triangulated category $\mathcal{D}/\mathcal{C}$ may have many more hearts. The next definition encodes the key restriction on the pairs of hearts and subcategories we consider. The reader may compare with [BPPW22, Section 3] for other criteria for hearts (or slicings) to descend to quotient categories or to be lifted from there.

Definition 6.4. *Let $\mathcal{V}$ be a full thick triangulated subcategory of $\mathcal{D}$, and $\mathcal{H}$ a heart of $\mathcal{D}$. We say that $\mathcal{H}$ is $\mathcal{V}$-*compatible *if $\mathcal{H} \cap \mathcal{V}$ is a heart of $\mathcal{V}$ and it is a Serre full subcategory of $\mathcal{H}$.*

We now return to our case of interest, i.e., $\mathcal{D} = \mathcal{D}_3(\mathbf{S}_\Delta)$ and $\mathcal{V} = \mathcal{D}_3(\Sigma)$, for a choice of a collapse ν. We denote by π_ν the quotient functor

$$\pi_\nu : \mathcal{D} \twoheadrightarrow \mathcal{D}/\mathcal{V}, \tag{6.2}$$

and always consider its essential images.

Proposition 6.5. *Let $\mathbb{T}$ be any refinement of a partial triangulation $\mathbb{A}$ of $\overline{\mathbf{S}}_{\mathbf{w}}$. Then the canonical heart $\mathcal{H} = \mathcal{H}(\Gamma_{\mathbb{T}})$ is $\mathcal{V}$-compatible. Moreover, the quotient heart $\overline{\mathcal{H}} \simeq \mathcal{H}/(\mathcal{H} \cap \mathcal{V})$ is independent of the choice of the refinement.*

Proof. The first statement follows from the fact that $\mathcal{H}$ is finite and $\mathcal{H} \cap \mathcal{V}$ is also finite, generated by the simples corresponding to arcs in $\mathbb{T} \setminus \mathbb{A}$.

The second statement follows from combining Proposition 5.10 and Lemma 6.6 (below). More precisely, there is a sequence of flips/mutation connecting different refinements $\mathbb{T}_1$ and $\mathbb{T}_2$ of the partial triangulation $\mathbb{A}$ by Proposition 5.10 which gives the same quotient hearts of $\mathcal{H}(Q_{\mathbb{T}_1})$ and $\mathcal{H}(Q_{\mathbb{T}_2})$ by Lemma 6.6. $\square$

Let $I \subseteq Q_0 = \{1, \ldots, n\}$ be a non-empty subset, and $\mathbf{i} = (i_l^{\epsilon_l}, \ldots, i_1^{\epsilon_1})$ be an ordered sequence with $i_j \in I$ and $\epsilon_j \in \{\sharp, \flat\}$. By the simple tilting formula (2.3), the sequence $\mathbf{i}$ induces a sequence of simple tiltings

$$\mu_{\mathbf{i}}\mathcal{H} = \mu_{i_l}^{\epsilon_l} \cdots \mu_{i_1}^{\epsilon_1}\mathcal{H}$$

for any heart $\mathcal{H}$ whose simples are parameterized by Q_0.

Lemma 6.6. *The quotient heart is an invariant under simple tilting in I, in the sense that for any $\mathbf{i} = (i_l^{\epsilon_l}, \ldots, i_1^{\epsilon_1})$ as above*

$$\mathcal{H}(Q,W)/\mathcal{H}\left((Q,W)_{|I}\right) \;=\; \mu_{\mathbf{i}}\mathcal{H}\left(Q,W\right)/\mu_{\mathbf{i}}\mathcal{H}\left((Q,W)_{|I}\right). \tag{6.3}$$

Proof. We only consider the case of a single mutation corresponding to a simple tilt, i.e. $\mathbf{i} = i^{\epsilon}$ for $i \in I$. Possibly repeating the argument then proves the statement. Let $\operatorname{Sim}\mathcal{H}(Q,W) = \{S_k \mid k \in Q_0\}$. Take $\mathcal{D} = \operatorname{pvd}(Q,W)$ and $\mathcal{V} = \operatorname{pvd}(Q,W)_I$ together with the hearts $\mathcal{H} = \mathcal{H}(Q,W)$ and $\mathcal{H}' = \mu_{S_i}^{\epsilon}\mathcal{H}$. Applying Lemma 6.7 below we obtain the lemma. $\square$

Lemma 6.7. *Suppose that $\mathcal{H}, \mathcal{H}'$ are $\mathcal{V}$-compatible hearts of bounded t-structures in $\mathcal{D}$. Then the bounded t-structures induced on the quotient $\mathcal{D}/\mathcal{V}$ by a twice t-exact fully faithful functor $\iota : \mathcal{V} \to \mathcal{D}$,*

$$(\mathcal{V}, \mathcal{H} \cap \mathcal{V}) \to (\mathcal{D}, \mathcal{H}) \;\; \textit{and}$$
$$(\mathcal{V}, \mathcal{H}' \cap \mathcal{V}) \to (\mathcal{D}, \mathcal{H}'),$$

coincide if $\mathcal{H}' = \mu_{\mathcal{T}}\mathcal{H}$ at some torsion class $\mathcal{T} \subset \mathcal{H}$ such that $\mathcal{T} \subset \mathcal{V}$.

An analogous statement holds, and can be proven similarly, for $\mathcal{H}' = \Phi_S\mathcal{H}$ for $S \in \mathcal{V}$, where Φ_S is the spherical twist at the simple $S \in \mathcal{H}$.

Proof. Let $\mathcal{F} := \mathcal{T}^{\perp}$ in $\mathcal{H}$ and $\mathcal{H}' = \mathcal{F} \perp \mathcal{T}[-1]$ be the $\mathcal{T}$-tilted heart in $\mathcal{D}$. We consider the diagram

$$\begin{tikzcd}
\mathcal{H}_{\mathcal{V}} = \mathcal{V} \cap \mathcal{H} \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{H}/\mathcal{H}_{\mathcal{V}} \\
\bigcap & \bigcap & \\
\mathcal{V} \arrow[r, "\iota"] & \mathcal{D} \arrow[r, "\pi"] & \mathcal{D}/\mathcal{V} \\
\bigcup & \bigcup & \\
\mathcal{H}'_{\mathcal{V}} = \mathcal{V} \cap \mathcal{H}' \arrow[r] & \mathcal{H}' \arrow[r] & \mathcal{H}'/\mathcal{H}'_{\mathcal{V}}
\end{tikzcd}$$

Serre-ness of $\mathcal{H}_{\mathcal{V}} \subset \mathcal{H}$, $\mathcal{H}_{\mathcal{V}}{}' \subset \mathcal{H}'$ is equivalent to π being $\mathcal{H}$- and $\mathcal{H}'$-exact, and $\mathcal{D}/\mathcal{V}$ is endowed with bounded t-structures with hearts $\pi(\mathcal{H})$, $\pi(\mathcal{H}')$ (Proposition 6.2).

By hypothesis, $\mathcal{T} \subset \mathcal{H}_{\mathcal{V}}, \mathcal{T}[-1] \subset \mathcal{H}_{\mathcal{V}}'$ are in the kernel of the quotient functor. Moreover, by definition of a torsion pair, for any $E \in \mathcal{H}$, there are $T \in \mathcal{T}$, $F \in \mathcal{F}$, and a short exact sequence

$$0 \to T \to E \to F \to 0,$$

that yields to a short exact sequence in the quotient $\pi(\mathcal{H})$

$$0 \to \pi(T) \simeq 0 \to \pi(E) \to \pi(F) \to 0.$$

This means that for any $\pi(E) \in \pi(\mathcal{H})$ there is $\pi(F) \in \pi(\mathcal{F})$ such that $\pi(E) \simeq \pi(F)$ in $\pi(\mathcal{H})$. Similarly, for any $E' \in \mathcal{H}'$, there is $G \in \mathcal{F}$ such that $\pi(E') \simeq \pi(G)$ in $\pi(\mathcal{H}')$. Hence the fully faithful functor $\pi(\mathcal{F}) \to \pi(\mathcal{H})$ is also essentially surjective. Therefore, $\pi(\mathcal{F}) \simeq \pi(\mathcal{H})$, and similarly $\pi(\mathcal{F}) \simeq \pi(\mathcal{H}')$. We conclude that the bounded t-structures on the quotient $\mathcal{D}/\mathcal{V}$ induced by $\mathcal{H}$ and $\mathcal{H}'$ coincide. □

6.3. **The exchange graphs of hearts of quotient type and its principal part.** Consider the exchange graph $\mathrm{EG}(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ of the quotient category. We want to relate this exchange graph to the exchange graph of $\mathcal{D}(\mathbf{S}_\Delta)$. We define the *principal part* $\mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ to be full subgraph of $\mathrm{EG}(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ whose vertices can be realized as quotients of $\mathcal{V}$-compatible hearts $\mathcal{H} \in \mathrm{EG}^\circ(\mathcal{D}(\mathbf{S}_\Delta))$ in a fixed connected component. As in the topological situation in Section 5.4, it is a priori not clear that the principal part consists of connected components of $\mathrm{EG}(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$. The next proposition prepares to show that this is indeed the case, namely that simple tilts of quotient hearts stem from simple tilts of $\mathcal{H}$, if the heart $\mathcal{H}$ is conveniently chosen in terms of some Ext-condition:

Proposition 6.8. *Suppose that $\mathcal{H}$ is finite rigid heart in $\mathcal{D}$ with an abelian subcategory $\mathcal{K}$ such that $\operatorname{Sim}\mathcal{K} \subset \operatorname{Sim}\mathcal{H}$. Denote by $\mathcal{V} = \operatorname{thick}(\mathcal{K})$. Let $S \in \operatorname{Sim}\mathcal{H} \setminus \operatorname{Sim}\mathcal{K}$ satisfying $\operatorname{Ext}^1(\mathcal{K}, S) = 0$. Then the simple tilting $\mathcal{H} \xrightarrow{S} \mathcal{H}^\sharp_S$ in $\mathcal{D}$ induces a simple tilting of quotient hearts*

$$\overline{\mathcal{H}} \xrightarrow{\overline{S}} \overline{\mathcal{H}}^\sharp_{\overline{S}} = \overline{\mathcal{H}^\sharp_S}$$

in $\mathcal{D}/\mathcal{V}$, where $\overline{?}$ is the essential image of $?$ under $\mathcal{D} \twoheadrightarrow \mathcal{D}/\mathcal{V}$.

Proof. As $\mathcal{H}$ and $\mathcal{K}$ are both abelian and finite, $\mathcal{K}$ is Serre in $\mathcal{H}$ and the image of any simple T in $\operatorname{Sim}\mathcal{H} \setminus \operatorname{Sim}\mathcal{K}$ is a simple $\overline{T}$ in $\mathcal{H}/\mathcal{K}$. By the simple tilting formula (2.3), simples in $\operatorname{Sim}\mathcal{K}$ remains in $\operatorname{Sim}\mathcal{H}^\sharp_S$ by the Ext^1-vanishing property of S. Thus, $\mathcal{K}$ is also an abelian Serre subcategory of $\mathcal{H}^\sharp_S$. By Proposition 6.2, the hearts $\mathcal{H}$ and $\mathcal{H}^\sharp_S$ induce two hearts $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}^\sharp_S}$ in $\mathcal{D}/\mathcal{V}$, such that the quotient functor π is t-exact. The t-structures on the quotient are the images of the t-structures on $\mathcal{D}$, therefore $\mathcal{H} \leq \mathcal{H}^\sharp_S \leq \mathcal{H}[1]$ implies

$$\overline{\mathcal{H}} \leq \overline{\mathcal{H}^\sharp_S} \leq \overline{\mathcal{H}}[1].$$

Moreover, $\langle S \rangle = \mathcal{H}^\sharp_S[-1] \cap \mathcal{H}$ implies

$$\langle \overline{S} \rangle \;=\; \overline{\mathcal{H}^\sharp_S}[-1] \cap \overline{\mathcal{H}}.$$

By Lemma 2.1, we see that $\overline{\mathcal{H}^\sharp_S}$ is indeed a forward tilt of $\overline{\mathcal{H}}$ with respect to the simple $\overline{S}$. □

We can now state the generalization of Theorem 5.9 to non-simple weights.

Theorem 6.9. *Fix a triangulation* $\mathbb{T}_0$ *and the component of* $\mathrm{EG}^\circ(\mathcal{D}(\mathbf{S}_\Delta))$ *corresponding to* $\mathrm{pvd}(\Gamma_{\mathbb{T}_0})$*. There is an isomorphism*

$$\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w}) \cong \mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$$

of the principal parts, determined by $\mathbb{T}_0$ *and* $\mathrm{EG}^\circ(\mathcal{D}(\mathbf{S}_\Delta))$ *respectively, of the exchange graphs for partial triangulations and for hearts of quotient type.*

In particular $\mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ *is a union of connected components of* $\mathrm{EG}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$*.*

Proof. Let $\mathbb{A}$ be a partial triangulation in $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ with a refinement $\mathbb{T}$ in the component of $\mathbb{T}_0$. Define $\varphi : \mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w}) \to \mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ on vertices by mapping $\mathbb{A}$ to the quotient $\overline{\mathcal{H}}(\mathbb{T})$ of the canonical heart $\mathcal{H}(\mathbb{T})$ of $\mathrm{pvd}(\Gamma_\mathbb{T})$ in $\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}) = \mathcal{D}/\mathcal{V}$. This is well-defined by Proposition 6.5. The surjectivity of φ follows from the surjectivity part of the isomorphism (5.7) from Theorem 5.9. For the injectivity of φ we combine the injectivity part of this isomorphism with Proposition 5.10 and Proposition 6.5.

We now consider the edges. For any forward flip $\mathbb{A} \xrightarrow{\gamma} \mathbb{A}^\sharp_\gamma = \mathbb{A}'$ in $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$, by Proposition 5.13, we can refine it to a forward flip $\mathbb{T} \xrightarrow{\gamma} \mathbb{T}^\sharp_\gamma = \mathbb{T}'$ in $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$ with the property that there is no arrow from γ to any open arc in $\mathbb{T} \setminus \mathbb{A}$ in $Q_\mathbb{T}$. Let $\mathcal{H}(\mathbb{T}) \xrightarrow{S} \mathcal{H}(\mathbb{T}')$ be the simple tilting corresponding to $\mathbb{T} \xrightarrow{\gamma} \mathbb{T}'$, i.e., so that the simple S corresponds to the arc γ. Let $\mathcal{K}$ be the subcategory of $\mathcal{H}(\mathbb{T})$ generated by the simples in $\mathrm{Sim}\,\mathcal{H}(\mathbb{T})$ corresponding to arcs in $\mathbb{T} \setminus \mathbb{A}$. By [KY11, Lemma 2.15] the no-arrow-condition above implies $\mathrm{Ext}^1(\mathcal{K}, S) = 0$. Then by Proposition 6.8 the simple tilting at S induces a ('quotient') simple tilting $\mathcal{H}(\mathbb{A}) \to \mathcal{H}(\mathbb{A}')$, and this is indeed an edge in $\mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$. Conversely, every edge in $\mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ arises by definition (and (5.7)) from a flip $\mathcal{H}(\mathbb{T}) \xrightarrow{S(\gamma)} \mathcal{H}(\mathbb{T}')$ between triangulations in the component of $\mathbb{T}_0$. This gives rise by definition to an edge in $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$. We have thus shown that φ is indeed a graph isomorphism.

For the last statement recall from the end of the proof of Proposition 5.13 that $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w}) \cong \mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ is an (m,m)-regular graph, where m is the number of edges in any $\mathbf{w}$-mixed-angulation of $\overline{\mathbf{S}}_\mathbf{w}$. On the other hand, $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$ has at most $m = \mathrm{rank}(K(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})))$ many edges. Since $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ is defined as a (full) subgraph of $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$, there cannot be any edges of $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ connecting a vertex of $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ to a vertex outside this subgraph. It must thus consist of components of $\mathrm{EG}(\overline{\mathbf{S}}_\mathbf{w})$, as we claimed. □

6.4. **The symmetry groups.** We study the symmetry groups of the surfaces and the categories, which will be used later. For $\mathcal{D} = \mathcal{D}_3(\mathbf{S}_\Delta)$, we have the following subgroups

$$\mathrm{Nil}^\circ(\mathcal{D}) \subset \mathrm{Aut}^\circ_K(\mathcal{D}) \subset \mathrm{Aut}^\circ(\mathcal{D}) \subset \mathrm{Aut}(\mathcal{D}) \tag{6.4}$$

defined as follows. $\mathrm{Aut}^\circ(\mathcal{D})$ is the subgroup of $\mathrm{Aut}(\mathcal{D})$ consisting on autoequivalences of $\mathcal{D}$ that preserve the principal component $\mathrm{Stab}^\circ(\mathcal{D})$ corresponding to $\mathrm{EG}^\circ(\mathcal{D})$. Let $\mathrm{Aut}^\circ_K(\mathcal{D})$ be the subgroup of autoequivalences that moreover act as identity on the Grothendieck group $K(\mathcal{D})$. We call autoequivalences that act trivially on $\mathrm{Stab}^\circ(\mathcal{D})$ *negligible autoequivalences*. We will also be interested in the quotients

$$\mathcal{A}ut^\circ(\mathcal{D}) = \mathrm{Aut}^\circ(\mathcal{D})/\,\mathrm{Nil}^\circ(\mathcal{D}) \quad \text{and} \quad \mathcal{A}ut^\circ_K(\mathcal{D}) = \mathrm{Aut}^\circ_K(\mathcal{D})/\,\mathrm{Nil}^\circ(\mathcal{D})\,. \tag{6.5}$$

Note that as $\mathcal{A}ut^\circ(\mathcal{D})$ acts faithfully on $\mathrm{Stab}^\circ(\mathcal{D})$, it also acts faithfully on $\mathrm{EG}^\circ(\mathcal{D})$.

As preparation, we show that autoequivalences correspond to mapping classes in the classical case. The following result is implicit in [KQ20]:

Proposition 6.10. *There is an embedding*

$$i_{\mathbb{T}_0} \colon \mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_\Delta)) \to \mathrm{MCG}(\mathbf{S}_\Delta)$$

depending on the choice of the initial triangulation $\mathbb{T}_0$. *Restricted to* $\mathrm{ST}(\Gamma_{\mathbb{T}_0})$, *the embedding* $i_{\mathbb{T}_0}$ *becomes the isomorphism between twist groups in Theorem 5.9.*

The map $i_{\mathbb{T}_0}$ *surjects onto the subgroup* $\mathrm{MCG}^\circ(\mathbf{S}_\Delta)$ *of* $\mathrm{MCG}(\mathbf{S}_\Delta)$ *that stabilizes the component* $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$.

Proof. Given $f \in \mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_\Delta))$, it maps the heart $\mathcal{H}_0$ associated to $\mathbb{T}_0$ to some heart $\mathcal{H} \in \mathrm{EG}^\circ(\mathcal{D}_3(\mathbf{S}_\Delta))$. Let $\mathbb{T}$ be the triangulation corresponding to $\mathcal{H}$ in (5.7). Since f is an autoequivalence, there is an element $\gamma \in \mathrm{MCG}(\mathbf{S})$ that maps the triangulation T of the corresponding undecorated surface to T_0. In fact, in this way [BS15, Theorem 9.9] (see also [KQ20, Theorem 4.12]) show that there is short exact sequence

$$1 \to \mathcal{ST}(\mathcal{D}) \to \mathcal{A}ut^\circ(\mathcal{D}) \to \mathrm{MCG}(\mathbf{S}) \to 1, \tag{6.6}$$

where $\mathcal{ST}(\mathcal{D}) = \mathcal{ST}(\mathcal{D}_3(\mathbf{S}_\Delta)) = \mathcal{ST}(\Gamma_{\mathbb{T}_0})$ is the image of the spherical twist group in the quotient by negligible autoequivalences. It thus suffice to alter γ by an element in the surface braid group to exhibit an element an element $i_{\mathbb{T}_0}(f)$ in $\mathrm{MCG}(\mathbf{S}_\Delta)$ that maps $\mathbb{T}_0$ to $\mathbb{T}$. This element is unique up to isotopy by the Alexander Lemma (stating that any homeomophism of a once-decorated disk is isotopy to identity if it preserves the boundary pointwise). This uniqueness also shows that the assignment $i_{\mathbb{T}_0}(\cdot)$ is actually a group homomorphism. It is injective as we have taken the quotient by the negligible autoequivalences. Comparing with the proof of Theorem 5.9 we see that $i_{\mathbb{T}_0}$ gives the isomorphism between twist groups there.

For the surjectivity and thanks to (6.6) we only need to ensure that the elements in the surface braid group that stabilizes $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$ are in the image of $i_{\mathbb{T}_0}(f)$. This stabilizer subgroup is the braid twist group $\mathrm{BT}(\mathbb{T}_0)$ by Theorem 5.8 and then the isomorphism $\mathrm{BT}(\mathbb{T}_0) \cong \mathrm{ST}(\Gamma_{\mathbb{T}_0})$ yields the claim, since the latter group is obviously a subgroup of $\mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_\Delta))$. ☐

Now let us consider the case of $\overline{\mathbf{S}}_{\mathbf{w}}$ obtained from $\mathbf{S}_\Delta$ by collapsing Σ, and the quotient categories $\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}) = \mathcal{D}_3(\mathbf{S}_\Delta)/\mathcal{D}_3(\Sigma)$. We need the follow subgroups of mapping class groups. For any subgroup G of $\mathrm{MCG}(\mathbf{S}_\Delta)$ let

$$G^\Sigma \;=\; \{g \in G \mid g(\Sigma) = \Sigma\}$$

be the subgroups leaving invariant the subsurface Σ. Finally we let $\underline{\mathrm{MCG}}(\Sigma)$ be the mapping class group of the unmarked surface associated with Σ and let $\underline{\mathrm{MCG}}^\circ(\Sigma) = \underline{\mathrm{MCG}}(\Sigma) \cap \mathrm{MCG}^\circ(\mathbf{S}_\Delta)^\Sigma$. We define the *liftable subgroup* of the mapping class group of the collapsed surface to be the quotient group and the subgroup

$$\mathrm{MCG}^\bullet_{\mathrm{lift}}(\overline{\mathbf{S}}_{\mathbf{w}}) \colon = \frac{\mathrm{MCG}^\circ(\mathbf{S}_\Delta)^\Sigma}{\underline{\mathrm{MCG}}^\circ(\Sigma)} \subseteq \mathrm{MCG}_{\mathrm{lift}}(\overline{\mathbf{S}}_{\mathbf{w}}) := \frac{\mathrm{MCG}(\mathbf{S}_\Delta)^\Sigma}{\underline{\mathrm{MCG}}(\Sigma)} \quad \subseteq \mathrm{MCG}(\overline{\mathbf{S}}_{\mathbf{w}})$$

Collapsing Σ, a subsurface with two non-isomorphic connected components, say one of them a disc and one with positive genus, such that the collapse results in the same weights $w_i > 1$, shows that in general the inclusion is strict: mapping class group elements that swap the marked points corresponding to the higher weights are not liftable.

We define the groups of autoequivalences like $\mathrm{Aut}^\bullet(\mathcal{D}), \mathcal{A}ut^\bullet(\mathcal{D})$ as in (6.4) and (6.5) by the requirement to stabilize the principal part (instead of a fixed component). For any subgroup $G \subset \mathcal{A}ut(\mathcal{D}_3(\mathbf{S}_\Delta))$ we write G^Σ for the subgroup that stabilizes the subcategory $\mathcal{D}_3(\Sigma)$. Finally we let $\underline{\mathcal{A}ut}^\circ(\mathcal{D}_3(\Sigma))$ be the subgroup of $\mathcal{A}ut^\circ(\mathcal{D}_3(\Sigma))$ consisting of elements that are restricted from elements in $\mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_\Delta))$. We define

$$\mathcal{A}ut^\bullet_{\mathrm{lift}}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})) \;=\; \frac{\mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_\Delta))^\Sigma}{\underline{\mathcal{A}ut}^\circ(\mathcal{D}_3(\Sigma))}\,. \tag{6.7}$$

We can now state the goal of this subsection:

Proposition 6.11. *There is an embedding*

$$i_{\mathbb{T}_0}\colon \mathcal{A}ut^\bullet_{\mathrm{lift}}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})) \to \mathrm{MCG}(\overline{\mathbf{S}}_\mathbf{w})$$

depending on the choice of the initial triangulation $\mathbb{T}_0$ of $\mathbf{S}_\Delta$. The map $i_{\mathbb{T}_0}$ surjects onto the subgroup $\mathrm{MCG}^\bullet_{\mathrm{lift}}(\overline{\mathbf{S}}_\mathbf{w})$.

To *prove Proposition 6.11* we only need the following two lemmas.

Lemma 6.12. *There is an isomorphism $\underline{\mathcal{A}ut}^\circ(\mathcal{D}_3(\Sigma)) \cong \underline{\mathrm{MCG}}^\circ(\Sigma)$ obtained by restriction of the isomorphism $\mathcal{A}ut^\circ(\mathcal{D}_3(\Sigma)) \to \mathrm{MCG}^\circ(\Sigma)$.*

Proof. Choose any triangulation $\mathbb{T}_0$ of $\mathbf{S}_\Delta$ that can be homotoped to a triangulation $\mathbb{T}_\Sigma$ of Σ. By definition, an element in $\underline{\mathcal{A}ut}^\circ(\mathcal{D}_3(\Sigma))$ is restricted from an element in $\gamma \in \mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_\Delta))^\Sigma$. Regarding this element γ as mapping class on S_Δ by Proposition 6.10, the restriction condition on the categorical side translates by Lemma 6.14 to the condition that the mapping class needs to preserve the collapsing data. Thus $i_{\mathbb{T}_0}(\gamma) \in \mathrm{MCG}^\circ(\mathbf{S}_\Delta)^\Sigma$ and by definition the initial automorphism $i_{\mathbb{T}_\Sigma}$ restricts to an injection $\underline{\mathcal{A}ut}^\circ(\mathcal{D}_3(\Sigma)) \to \underline{\mathrm{MCG}}^\circ(\Sigma)$. It is surjective since any element in $\underline{\mathrm{MCG}}^\circ(\Sigma)$ can be regarded as an element in $\mathrm{MCG}^\circ(\mathbf{S}_\Delta)$ or equivalently via $i_{\mathbb{T}_0}$ as an element in $\mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_\Delta))$. Restricted to $\mathcal{D}_3(\Sigma)$, we see that it is indeed an element in $\underline{\mathcal{A}ut}^\circ(\mathcal{D}_3(\Sigma))$ and the lemma follows. □

Lemma 6.13. *There is an isomorphism $\mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_\Delta))^\Sigma \to \mathrm{MCG}^\circ(\mathbf{S}_\Delta)^\Sigma$ obtained by restriction of the isomorphism $\mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_\Delta)) \to \mathrm{MCG}^\circ(\mathbf{S}_\Delta)$.*

Proof. We regard $\mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_\Delta))^\Sigma$ as a subgroup of $\mathrm{MCG}^\circ(\mathbf{S}_\Delta)$ and the condition of stabilizing $\mathcal{D}_3(\Sigma)$ translates topologically to stabilizing all simple closed arcs in Σ, using the correspondence between closed arcs and reachable spherical objects in [Qiu16, Thm. 6.6]. By [Qiu16, Lemma 4.6], Σ is in fact a neighbourhood of the union of any triangulation dual to the closed arcs in Σ. Thus, the condition of stabilizing the arcs is topologically equivalent to the condition of stabilizing Σ and the lemma follows. □

In the proofs above we have been using the following statement.

Lemma 6.14. *Let Σ_1 and Σ_2 be two DMS with simple weights and without punctures, with associated CY_3 categories $D_3(\Sigma_i)$. Then $D_3(\Sigma_1)$ is triangle equivalent to $D_3(\Sigma_2)$ if and only if Σ_1 is homeomorphic to Σ_2.*

Proof. The existence of a homeomorphism obviously implies the existence of a triangle equivalence. For the converse we reconstruct the surface Σ from a single heart $\mathcal{H}$ in a way that is obviously inverse to the construction of $\mathcal{H}$ from Σ. First, the quiver Q is the graph given by the simples S_i in $\mathcal{H}$ with edges given by non-trivial Ext^1's. Second we reconstruct the potential W from the Ext-algebra of the Γ-module $S = \oplus_{i\in\mathrm{Sim}(\mathcal{H})} S_i$. This Ext-algebra carries an A_∞-structure, unique up to A_∞-isomorphism. An explicit construction of this structure is given for any quiver with potential in [Kel11a, Appendix A.15]. Since in our case the potential consists of 3-cycles only, the Ext-algebra is formal, i.e. the higher multiplication maps m_n for $n \geq 2$ vanish. This means that the model given in loc. cit. is the minimal model of the A_∞-isomorphism class and that the multiplication map m_2 given in log. cit. is canonically associated with S. This map m_2 determines the potential W uniquely.

Finally, we reconstruct the surface from (Q, W), reversing the construction in Section 3.3: For 3-cycle in W glue a triangle to the corresponding edges of Q. For each arrow of Q not in a 3-cycle, glue a triangle with one edge as boundary edge to the two edges representing head and tail of the arrow. For each vertex of Q to which only a single of the preceding rules apply (in the since that such a 3-cycle passes through the vertex, or such an arrow starts or ends in the vertex), glue a triangle with two boundary edges (and one boundary marked point). Finally, if for a vertex none of the preceding rules apply (which happens only for the A_1-quiver), then we glue two such triangles. Since all data used for this reconstruction procedure are preserved by the equivalence, the lemma follows. □

7. An extension of the Bridgeland-Smith correspondence

In their paper [BS15] Bridgeland and Smith gave a correspondence roughly between the space of framed quadratic differentials with only simple zeroes and stability conditions on the category $\mathrm{pvd}(Q, W)$ where (Q, W) is the quiver with potential associated with a saddle-free differential. In this section we recall their result and extend it to our main result, a correspondence between the space of framed quadratic differentials with higher order zeros and certain stability conditions supported on the quotient categories introduced in Section 6. Various mapping class subgroups and groups of autoequivalence have been defined in Section 6.4.

7.1. **The original Bridgeland-Smith correspondence.** We state the Bridgeland-Smith correspondence in the version of [KQ20] lifted to Teichmüller-framed quadratic differentials and in the case that each boundary component of $\mathbb{S}$ has at least one marked point, i.e. the quadratic differentials have poles of higher order ≥ 3 only. In this way we avoid the extra technicalities of local orbifold structure (the space $\mathrm{Quad}_\heartsuit(\mathbb{S}, \mathbb{M})$ introduced in [BS15]). For the notation concerning spaces of quadratic differentials we refer the reader to Section 4.1.

Fix a genus g polar part $\mathbf{w}^-$ of the signature, the number $n = 2g - 2 + |\mathbf{w}^-|$ of simple zeros with $\mathbf{S}_\Delta$ a reference surface of this type, and fix an initial Teichmüller-framed quadratic differential $(X_0, q_0, \psi_0) \in \mathrm{FQuad}(\mathbf{S}_\Delta)$ and suppose that q_0 is saddle-free. Denote by $\mathrm{FQuad}^\circ(\mathbf{S}_\Delta)$ the connected component containing q_0. Using (the classical version in [BS15] for simple weights of) Definition 4.7 the differentials gives us a triangulation $\mathbb{T}_0$, which gives a quiver with potential (Q_0, W_0) by the construction in Section 3.3 and thus the category $\mathcal{D} = \mathrm{pvd}(\Gamma_{\mathbb{T}_0})$ defined in Section 2.3 with its standard heart $\mathcal{H}_0$. Fix a canonical double cover $(\widehat{X_0}, \omega_0)$. For each horizontal strip let η_i be the saddle connection crossing that strip, by definition a closed arc dual to $\mathbb{T}_0$. Let $\widehat{\eta_i}$ be the corresponding hat-homology class, oriented such that its ω_0-period $\mathrm{Per}(\widehat{\eta_i}) \in \overline{\mathbb{H}}$. Denoting by $S_i \in \mathrm{Sim}(\mathcal{H}_0)$ the simple object corresponding to $\widehat{\eta}_i$ and define the map Z_0 by $Z_0(S_i) = \mathrm{Per}(\widehat{\eta}_i)$. In total we defined a stability condition $\sigma_0 = (\mathcal{H}_0, Z_0)$.

Fix an isomorphism $\theta_0 : \Gamma \to \widehat{H}_1(q_0)$ and fix an isomorphism $\nu_0 : \Gamma \to K(\mathcal{D})$. Recording just the central charge gives a map

$$\begin{aligned} \pi_2 : \mathrm{Stab}^\circ(\mathcal{D}) &\to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}), \\ (Z, \mathcal{A}) &\mapsto (Z \circ \nu_0). \end{aligned} \tag{7.1}$$

whose factorization through $\mathrm{Stab}^\circ(\mathcal{D})/\mathcal{A}ut_K(\mathcal{D})$ we denote by the same symbol. On the other hand, on the space of period-framed quadratic differentials the projection gives a map

$$\begin{aligned} \pi_1 : \mathrm{Quad}_g^\Gamma(1^r, \mathbf{w}^-) &\to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}), \\ (q, \rho) &\mapsto (\mathrm{Per}(q) \circ \rho \circ \theta_0). \end{aligned} \tag{7.2}$$

Note that our notion of Teichmüller framing does not frame the double cover, while the hat-homology depends on the double cover. Thus a priori it is not clear whether the cover $\mathrm{FQuad}(\mathbf{S}_\Delta)$ dominates the cover $\mathrm{Quad}_g^\Gamma(1^r, \mathbf{w}^-)$. This is proven along with the following theorem.

Theorem 7.1. *There is an isomorphism of complex manifolds*

$$K : \mathrm{FQuad}^\circ(\mathbf{S}_\Delta) \to \mathrm{Stab}^\circ(\mathcal{D})\,. \tag{7.3}$$

The natural covering map $\mathrm{FQuad}(\mathbf{S}_\Delta) \to \mathrm{Quad}_g(1^r, \mathbf{w}^-)$ *factors through a covering* $\pi_0 : \mathrm{FQuad}(\mathbf{S}_\Delta) \to \mathrm{Quad}_g^\Gamma(1^r, \mathbf{w}^-)$. *The map K commutes with the maps $\pi_1 \circ \pi_0$ and π_2 to* $\mathrm{Hom}(\Gamma, \mathbb{C})$ *given by periods and by the central charge respectively. This map K is equivariant with respect to the action of the mapping class group* $\mathrm{MCG}(\mathbf{S}_\Delta)$ *on the domain and of the group* $\mathcal{A}ut^\circ(\mathcal{D})$ *on the range. The map K descends to isomorphisms of complex orbifolds*

$$\begin{aligned} K^\Gamma : \mathrm{Quad}_g^{\Gamma,\circ}(1^r, \mathbf{w}^-) &\to \mathrm{Stab}^\circ(\mathcal{D})/\mathcal{A}ut_K^\circ(\mathcal{D}) \\ \overline{K} : \mathrm{Quad}_g(1^r, \mathbf{w}^-) &\to \mathrm{Stab}^\circ(\mathcal{D})/\mathcal{A}ut^\circ(\mathcal{D})\,, \end{aligned} \tag{7.4}$$

where $\mathrm{Quad}_g^{\Gamma,\circ}(1^r, \mathbf{w}^-)$ *is the connected component given by the image of* π_0.

Proof. The existence of K^Γ is the content of [BS15, Theorem 11.2]. The map is constructed in Propositions 11.3 and 11.11 and the fact that the isomorphism descends is argued along with diagram (11.6) in loc. cit. The lift K is constructed in [KQ20, Theorem 4.13]. The quotient $\overline{K}$ is obtained from K thanks to Proposition 6.10. We expand on two arguments that are only briefly discussed in these sources. First, the orbifold structure requires that $\mathcal{A}ut^\circ(\mathcal{D})$ acts properly discontinuously. See the proof of Theorem 7.2 for this argument.

Second, we elaborate on the existence of π_0, implicitly needed in [KQ20, Theorem 4.13]. As we will recall in more detail in the proof of Theorem 7.2, the map K is constructed first as a map K_0 on the locus B_0 of saddle-free differentials proceeding as we did with q_0 above. This involves the choice of a lift $\widehat{\eta}_i$ of the crossing saddle connections η_i on each of the chambers, i.e., connected components of B_0. This lift also provides the map π_0 on each chamber. The map K_0 is then extended to a map K_2 on the locus B_2 of tame differentials, identifying chambers adjacent by forward flips and and forward tilts respectively, using the exchange graph isomorphism (5.7) in Theorem 5.9. The continuity of K_2 and π_0 on $B_2 \setminus B_0$ then follows once we checked the following condition, using our standard choice of oriented lifts of saddle connections so that their periods are $\overline{\mathbb{H}}$-valued: The lift of the

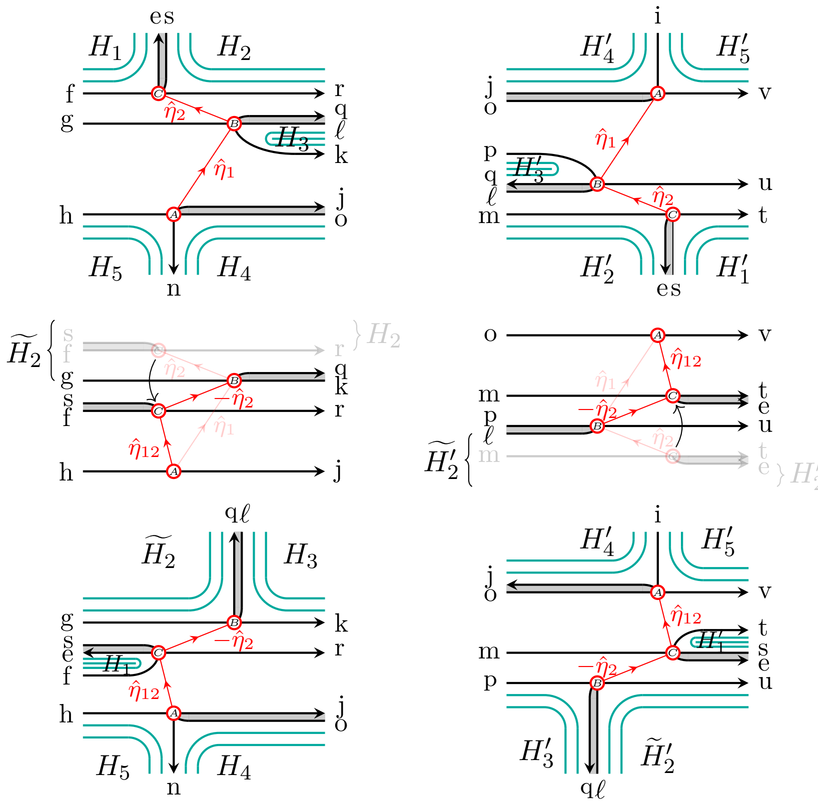

FIGURE 9. Hat-homology classes on the double cover before (first row) and after (last row) a flip. The middle row illustrates the transition.

flipped standard saddle connection are related to the lifts of the original standard saddle connection in the same way as the image simples objects are related, namely by (2.3)[2]. This is a local topological statement, to be checked only for the case where ext^1 is non-trivial. Consider Figure 9 where shaded slits indicate branch cuts and black arrows the positive real axis. In particular on the central horizontal strips the imaginary axis points upwards and all hat-homology classes are oriented to have positive imaginary part. We can now verify $\widehat{\eta_{12}} = \widehat{\eta_1} + \widehat{\eta_2}$ in hat-homology, using the obvious homotopy exhibiting this relation for each of the two sheets of the canonical double cover. $\square$

7.2. **The correspondence in the generalized setting.** We modify the setup of Section 7.1 as follows. Let now $\mathbf{w}$ be any tuple of non-zero integers and let $\overline{\mathbf{S}}_{\mathbf{w}}$ be a wDMS, obtained as collapse of the surface $\mathbf{S}_\Delta$. (That is, a collision $g(\overline{\mathbf{S}}_{\mathbf{w}}) = g(\mathbf{S}_\Delta)$ is the easiest possibility to realize this situation but we also allow the case where the collapse is not a collision.) Applying Definition 4.7 to an initial saddle-free $(X_0, q_0, \psi_0) \in \mathrm{FQuad}(\overline{\mathbf{S}}_{\mathbf{w}})$ now gives a mixed-angulation $\mathbb{A}_0$ on $\overline{\mathbf{S}}_{\mathbf{w}}$, and thus a

[2]This is probably well-known, see around [BS15, Proposition 10.4], but we are uncertain about the role of the orientation of lifts there.

partial triangulation on $\mathbf{S}_\Delta$, which we refine to an initial triangulation $\mathbb{T}_0$ on $\mathbf{S}_\Delta$. We let $\overline{\mathcal{H}}_0$ be the quotient heart in the quotient category $\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})$ given by the construction in Theorem 6.9. As above let η_i be the saddle connections crossing strips, and lift them to hat-homology classes $\widehat{\eta_i}$ using the convention in Section 3.2 and orient the lifts (which now might be non-closed, i.e., relative periods) so that $\mathrm{Per}(\widehat{\eta_i}) \in \overline{\mathbb{H}}$. Finally, as above we define the map Z_0 by $Z_0(S_i) = \mathrm{Per}(\widehat{\eta_i})$ and define the stability condition $\sigma_0 = (\overline{\mathcal{H}}_0, Z_0)$ on $\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})$.

The choice of q_0 and $\mathbb{T}_0$ above fixes a principal part $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ of the exchange graph of partial triangulations and by the isomorphism in Theorem 6.9 also a principal part in $\mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$. We let $\mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ be the connected components whose associated partial triangulations belong to $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$. These components include the one with (X_0, q_0, ψ), and possibly others. On the stability side we consider the set

$$\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})) \;=\; \mathbb{C}\cdot \bigcup_{\mathcal{H}\in \mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))} \mathrm{U}(\mathcal{H}). \tag{7.5}$$

which is not a priori a union of connected components of $\mathrm{Stab}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$.

For the period and central charge map we fix isomorphisms $\theta_0 : \Gamma \to \widehat{H}_1(q_0)$ and $\nu_0 : \Gamma \to K(\mathcal{D})$, keeping in mind that the rank of Γ depends on $(\mathbf{w}, \mathbf{w}^-)$. We define the projections π_1 and π_2 just as in (7.1) and (7.2), with domains $\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ and $\mathrm{Quad}_g^\Gamma(\mathbf{w}, \mathbf{w}^-)$ respectively. Again it is not a priori clear whether the cover $\mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ dominates the cover $\mathrm{Quad}_g^\Gamma(\mathbf{w}, \mathbf{w}^-)$.

Moreover, recall the definition of the liftable mapping class groups and autoequivalences from Section 6.4. We write

$$\mathrm{Quad}_g(\mathbf{w}^\Sigma, \mathbf{w}^-) = \mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_\mathbf{w}) / \mathrm{MCG}^\bullet_{\mathrm{lift}}(\overline{\mathbf{S}}_\mathbf{w}).$$

This space is a finite cover of the moduli space of quadratic differentials where the zeros can be permuted only if the realization of $\overline{\mathbf{S}}_\mathbf{w}$ as collapse allows this, i.e. if the permutation can be lifted to a mapping class element in $\mathbf{S}_\Delta$.

Theorem 7.2. *There is an isomorphism of complex manifolds*

$$K : \mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_\mathbf{w}) \to \mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))\,. \tag{7.6}$$

The natural covering map $\mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_\mathbf{w}) \to \mathrm{Quad}_g(\mathbf{w}, \mathbf{w}^-)$ *factors through a covering* $\pi_0 : \mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_\mathbf{w}) \to \mathrm{Quad}_g^\Gamma(\mathbf{w}, \mathbf{w}^-)$. *The map K commutes with the maps* $\pi_1 \circ \pi_0$ *and* π_2 *to* $\mathrm{Hom}(\Gamma, \mathbb{C})$ *given by periods and by the central charge respectively. This map K is equivariant with respect to the action of the mapping class group* $\mathrm{MCG}^\bullet_{\mathrm{lift}}(\overline{\mathbf{S}}_\mathbf{w})$ *on the domain and of the group* $\mathcal{A}ut^\bullet_{\mathrm{lift}}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ *on the range. The map K descends to isomorphisms of complex orbifolds*

$$\begin{aligned} K^\Gamma &: \mathrm{Quad}_g^{\Gamma,\bullet}(\mathbf{w}, \mathbf{w}^-) \to \mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))/\mathcal{A}ut^\bullet_K(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w})) \\ \overline{K} &: \mathrm{Quad}_g(\mathbf{w}^\Sigma, \mathbf{w}^-) \to \mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))/\mathcal{A}ut^\bullet_{\mathrm{lift}}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))\,, \end{aligned} \tag{7.7}$$

where $\mathrm{Quad}_g^{\Gamma,\bullet}(\mathbf{w}, \mathbf{w}^-)$ *are the connected components given by the image of* π_0.

This result has topological consequences for the principal parts:

Corollary 7.3. *The principal part* $\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$ *is a union of connected components of* $\mathrm{Stab}(\mathcal{D}(\overline{\mathbf{S}}_\mathbf{w}))$. *In particular these components in the image of $FQuad(S_w)$ are generic-finite in the sense of Definition 2.4.*

Proof of Theorem 7.2. We proceed similarly as in the proof of Proposition 11.3 in [BS15] using the stratification (4.5) by the number of horizontal saddle connections. Recall for this purpose the definition of B_p and $F_p = B_p \setminus B_{p-1}$ from Section 4.2.

The maps on the tame locus. We first define the map on the saddle-free locus

$$K_0 : B_0 \to \mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$$

associating with a framed differentials a stability condition just as we did with q_0 in the introductory paragraph of this section.

We check that this map continuously extends to a map K_2 the tame locus. By Corollary 5.4 any two neighboring chambers C, C' in B_0 are related by a forward flip of the partial triangulations $\mathbb{A} \to \mathbb{A}'$ at some arc γ. Consider a differential q on the wall between C and C'. We need to show the continuity of the extension of K_0 to 0 along the arc

$$\rho : t \mapsto e^{it} q \in \mathrm{FQuad}^\bullet(\overline{\mathbf{S}}_{\mathbf{w}}) \quad \text{for} \quad t \in (-\epsilon, \epsilon) \setminus \{0\}\,. \tag{7.8}$$

By the isomorphism of exchange graphs from Theorem 6.9 there is a tilt at some simple S that relates $K_0(C)$ to a neighboring chamber. We moreover want to show that this chamber is indeed $K_0(C')$. This follows since the assignment of stability conditions to partial triangulations is defined using a refinement to a triangulation and the exchange graph isomorphism is derived from the isomorphism (5.7) at the level of triangulations, i.e., there are refinements so that the forward tilt lifts to a tilt $\mathbb{T} \xrightarrow{\gamma} \mathbb{T}'$. Next we check the compatibility of the lifted hat-homology classes $\widehat{\eta}_i$ at this wall-crossing, i.e. that the lifted classes of saddle connections crossing cylinders satisfy the same base change relation as the corresponding simples, which is (2.3). We checked this in the proof of Theorem 7.1 along with Figure 9 from the refining triangulation and this continues to hold after setting to zero the classes of dual complementary arcs and the corresponding simples. Now we are in position to use the periods of the dual arcs in $\mathbb{T}$ and the corresponding simples as a coordinate system on a full neighborhood of the wall, see Definition 2.3 and [BS15, Lemma 7.9]. We conclude, since the periods of all dual arcs (and hence all simples) vary continuously along the arc ρ.

Extension to non-tame differentials. We now construct K_p defined on B_p inductively, assuming the existence of K_{p-1} to eventually obtain $K = K_k$, where k was defined as the maximal number of horizontal saddle connections. Just as in [BS15, Proposition 5.5] for any differential $q \in F_p$ any small rotation $e^{it} q \in B_{p-1}$ for $0 < |t| < \varepsilon$ and ε small enough. By induction on p and $\mathbb{C}$-equivariance the limiting generalized stability conditions

$$\sigma_\pm(q) \;=\; \lim_{t \to 0^+} K_{\rho, p-1}(e^{\pm it} q) \tag{7.9}$$

exist and we need to show that they coincide. The continuity of the $\mathbb{C}$-action on framed differentials and $\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ implies that the agreement of the limits has a locally constant answer. Since walls have ends on any connected component of F_p (Proposition 4.2), this agreement locus is all of F_p. We thus obtain eventually a map K defined everywhere.

Injectivity. Suppose that the differentials q_1 and q_2 have the same image $K(q_1) = K(q_2)$. Using the $\mathbb{C}$-action we may assume that both differentials lie in the interior of chambers. Using the injectivity of the exchange graph isomorphism in Theorem 6.9 shows that the chambers agree and since periods of crossing saddle connections are coordinates we conclude $q_1 = q_2$.

The surjectivity. of K is obvious from the surjectvity of the exchange graph isomorphism in Theorem 6.9 and the compatibility with the $\mathbb{C}$-action.

The maps π_0. has been defined locally on each chamber of B_0 by using the lift to hat-homology with periods in $\mathbb{H}$. On each wall crossing this assignment has been verified along with the continuity of K_2 to be compatible with the base change in the Grothendieck group $K(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$. By definition of spaces of stability conditions simple objects are labeled globally and in particular a basis of $K(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ can be chosen globally over $\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ (and in fact over $\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))/\mathcal{A}ut^\bullet_K(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$). This implies that the base change formula (2.3) is consistent over loops in $\mathrm{EG}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$, i.e., the product of the wall-crossing base changes along a closed loop is the identity. Since hat-homology is isomorphic to the Grothendieck group (say in the initial chamber) this implies that the corresponding base change in hat-homology is consistent over loops in $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_{\mathbf{w}})$. This shows that the local definition of π_0 gives well-defined function on B_2. We extend π_0 to a global function over higher B_k using the $\mathbb{C}$-action just as we did with K_2.

The compatibility of K with the projections. This compatibility with π_2 and $\pi_1 \circ \pi_0$ holds on the initial chamber by definition, on all the other chambers by construction of π_0 and globally, since all these maps are equivariant with respect to the $\mathbb{C}$-action.

Quotient orbifolds. In order to show that $\mathrm{Stab}^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))/\mathcal{A}ut^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ is an orbifold we need to show the properness of the action of $\mathcal{A}ut^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ and that this group acts with finite stabilizers. For properness we use that the $\mathbb{C}$-action, which commutes with automorphisms, to assume that the two points whose orbits we have to separate lie in the interior of a chamber. Since $\mathcal{A}ut^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ maps (open) chambers to chambers, properness is obvious if the two orbits are never in a common chamber. Otherwise we use that $\mathcal{A}ut^\bullet(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ preserves the integral lattice $\Gamma^\vee$ in $\mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$ and the fact that there is no infinite sequence in $\mathrm{GL}_d(\mathbb{Z})$, where $d = \mathrm{rank}(\Gamma)$, that fails to displace a small ball. This argument shows moreover the finiteness of stabilizers. (Compare with [Smi18, Lemma 3.3].)

Descending to K^Γ and to $\overline{K}$. Recall from Proposition 6.11 the existence of an isomorphism $\mathcal{A}ut^\bullet_{\mathrm{lift}}(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}})) \to \mathrm{MCG}^\bullet_{\mathrm{lift}}(\overline{\mathbf{S}}_{\mathbf{w}})$. The equivalence of K with respect to this isomorphism follows from the construction in Proposition 6.10 (using the same initial triangulation), since flipping arcs commutes with the mapping class group action. □

The following proof adapts the argument of Bridgeland-Smith in a way so that the complete description of the moduli space of stable objects of given class (see [BS15, Theorem 11.6]) can be avoided. We expect the analogue of this theorem to have more and more complicated case distinctions as the entries of $\mathbf{w}$ grow.

Proof of Corollary 7.3. The image of the holomorphic map K is obviously open, so we need to check that it is closed. Suppose that we have a one-parameter family $\sigma(t)$ in $\mathrm{Stab}^{\bullet}(\mathcal{D}(\overline{\mathbf{S}}_{\mathbf{w}}))$ with $\sigma(t) = K(q(t))$ in the image of the comparison isomorphism for $t \neq 0$. To show that $\sigma = \sigma(0)$ is also in the image we need by [BS15, Proposition 6.8] a lower bound for the lengths of saddle connections of $q(t)$ as $t \to 0$. We claim that for each $t \neq 0$ and for each saddle connection γ on $q(t)$ there is stable object $E \in \sigma(t)$ with mass $m(E) = |Z(\gamma)|$. We then obtain the lower bound of lengths as $t \to 0$, since the masses of stable objects in the limit $\sigma(0)$ are bounded thanks to the support property of the stability condition.

To prove the claim we may assume by rotation that $Z(\gamma) \in \mathbb{R}$. After a small rotation now γ becomes a standard saddle connection crossing a horizontal strip. (The nearby directions where this is not true have a saddle connection or a spiral domain, hence a saddle connection, and this exception set is countable.) The stability condition corresponding to the slightly rotated differential has a simple (and hence stable) object of class α. This property persists after undoing the small rotation. □

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A.B.: Dipartimento di Informatica - Settore Matematica, Università di Verona, Strada Le Grazie 15, 37134 Verona - Italy
Email address: anna.barbieri@univr.it

M.M.: Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6-8, 60325 Frankfurt am Main, Germany
Email address: moeller@math.uni-frankfurt.de

Y.Q.: Yau Mathematical Sciences Center and Department of Mathematical Sciences, Tsinghua University, 100084 Beijing, China. & Beijing Institute of Mathematical Sciences and Applications, Yanqi Lake, Beijing, China
Email address: yu.qiu@bath.edu

J.S.: Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6-8, 60325 Frankfurt am Main, Germany
Email address: so@math.uni-frankfurt.de