Chapter 1

Spaces of Bridgeland stability conditions in Representation Theory

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The space of Bridgeland stability conditions is a complex manifold that can be attached to a triangulated category, of which it encodes some homological properties. These notes are an introduction to this topic, with a focus on examples from Representation Theory, and review the example of the Bridgeland-Smith correspondence for some quiver categories.

1.1 Introduction

The notion of stability in the context of algebra and geometry is traditionally intepreted as a classification tool to gather objects (the stable or semi-stable ones) in well-behaved moduli spaces.

Stability conditions for triangulated categories were first introduced by Tom Bridgeland at the beginning of 2000 in [14]. One of the major features of this notion is that by definition it incorporates the possibility of considering the set of all stability conditions as a complex manifold, denoted $\mathrm{Stab}(\mathcal{D})$, attached to a triangulated category $\mathcal{D}$, of which it encodes some homological properties. Since its introduction, the space $\mathrm{Stab}(\mathcal{D})$ has played a role in algebraic geometry, representation theory, mirror symmetry, and some branches of mathematical physics, providing interesting synergies. While these spaces are unknown in many cases, there are examples that are quite well understood.

The goal of these notes is to give an introduction to spaces of stability conditions on triangulated categories –with a view towards module categories. As an example, we consider the space of stability conditions of a class of three-Calabi-Yau categories from quivers with potential, that are well known in representation and cluster theory. In the whole chapter, instead of giving entire proofs, we try to emphasize the main ideas and ingredients or give references.

The chapter is organized as follows. In Section 1.2, we recall the definition of Bridgeland stability conditions, as it is currently predominantly accepted. The space $\mathrm{Stab}(\mathcal{D})$ is introduced in Section 1.3, together with its main properties as a topological and complex manifold. We recall how the geometry of the space is controlled by

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bounded t-structures on the category, and we briefly mention some research directions that have received attention in the last decade, concerned with the stability manifold itself. Section 1.4 is aimed to review the computation, due to Bridgeland and Smith, of the space of stability conditions on some quiver categories, summarised in Theorem 1.22 as an isomorphism involving moduli spaces of framed quadratic differentials on a weigthed marked surface. This example is the most familiar to the author, and it is used to see in practice some of the ingredients from Section 1.3 and to give a hint on possible fruitful interactions between Bridgeland stability conditions and other moduli problems. The relevant categories and the necessary notions of quadratic differentials from the theory of flat surfaces are briefly recalled in the Appendix, for consistency.

The material presented in this article reflects the research interests of the author, and there are therefore many interesting aspects and directions that are not covered or mentioned. These include the problem of constructing stability conditions for derived categories of varieties, for which there is neverteless availability of surveys, and using stability structures as invariants for doing geometry or enriching a category. They also include the questions about defining moduli spaces of objects, enumerative theories associated with Bridgeland stability conditions, wall-crossing phenomena and geometry originating from them.

1.2 Stability conditions

1.2.1 Preliminary definitions

We fix here some notation that will be use through the section and the whole paper. k is an algebraically closed field, and any category is additive, k-linear, and essentially small. A short exact sequence (s.e.s.) $0 \to A \to B \to C \to 0$ in an abelian category is often represented as $A \to B \to C$, while a distinguished triangle in a triangulated category is represented either as $A \to B \to C \to A[1]$ or as an usual triangle. Given subcategories $\mathcal{H}_1, \mathcal{H}_2$ of an abelian or a triangulated category C, and a set of objects $\mathcal{B}$, we define the following subcategories

$$\begin{aligned}
\mathcal{H}_1 *_C \mathcal{H}_2 &:= \{M \in C \mid \exists \text{ s.e.s or triangle } T \to M \to F \text{ s.t. } T \in \mathcal{H}_1, F \in \mathcal{H}_2\}\,,\\
\mathcal{B}^{\perp_C} &:= \{C \in C : \mathrm{Hom}_C(B, C) = 0,\ \forall\, B \in \mathcal{B}\} \ \text{ and similarly } {}^{\perp_C}\mathcal{B}\,,\\
\mathcal{H}_1 \perp_C \mathcal{H}_2 &:= \mathcal{H}_1 * \mathcal{H}_2\,,\ \text{ if } \mathcal{H}_2 = \mathcal{H}_1^{\perp_C} \text{ and } \mathcal{H}_1 = {}^{\perp_C}\mathcal{H}_2\,.
\end{aligned}$$

We denote by $\langle \mathcal{B} \rangle_C$ the closure under extensions and possibly shifts of $\mathrm{Add}\,\mathcal{B}$ in C, and we say that it is the subcategory *generated* by $\mathcal{B}$. We will usually omit the subscript C.

Definition 1.1. *An abelian category is called of* finite length *if for any $E \in \mathcal{A}$ there is a finite sequence $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ such that all E_i/E_{i-1} are simple. It will be called* finite *if, moreover, it has a finite number of simple objects.*

We recall that the Grothendieck group of an abelian (resp., triangulated) category C is the group generated by the classes $[-]$ of isomorphism of objects in C, modulo relations induced by short exact sequences (resp., distinguished triangles):

$$A \to B \to C(\to A[1]) \quad \text{implies} \quad [B] = [A] + [C].$$

It is denoted $K(C)$. It is easy to verify that $[A[-1]] = -[A]$ when C is triangulated. If C is an abelian category and $A \in C$, the class $-[A]$ does not represent any object in C.

If a triangulated category $\mathcal{D}$ is Hom-finite, that is for any $E, F \in \mathcal{D}$ the vector space $\oplus_i \operatorname{Hom}_{\mathcal{D}}(E, F[i])$ is finite dimensional, the Euler form $\chi : K(\mathcal{D}) \times K(\mathcal{D}) \to \mathbb{Z}$ is defined by

$$\chi([E],[F]) = \sum_i (-1)^i \dim \operatorname{Hom}_{\mathcal{D}}(E, F[i]).$$

The notion of a t-structure for a triangulated category was introduced in [13] by A. Beilinson, J. Bernstein, P. Deligne (t-category), and refined to the notion of a slicing in [14] by T. Bridgeland. We are insterested here in *bounded* t-structures, which are non-degenerate t-structures modelled on the decomposition of the bounded derived category $\mathcal{D}^b(\mathcal{A})$ of an abelian category $\mathcal{A}$ into objects with only non-positive non-zero cohomology $H^i(E) = 0, i > 0$, only non-negative non-zero cohomology $H^i(E) \neq 0, i < 0$, and their extensions.

Definition 1.2. *A* bounded t-structure *on a triangulated category $\mathcal{D}$ is defined by a full subcategory $\mathcal{L} \subset \mathcal{D}$ (called the aile), closed under shift $\mathcal{L}[1] \subset \mathcal{L}$, such that*

$$\mathcal{D} = \mathcal{L} \perp \mathcal{L}^{\perp}, \quad \text{and moreover} \tag{1.2.1}$$

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{L}[i] \cap \mathcal{L}^{\perp}[j]. \tag{1.2.2}$$

The heart *of a bounded t-structure $\mathcal{L} \subset \mathcal{D}$ is the full subcategory $\mathcal{A} = \mathcal{L} \cap \mathcal{L}^{\perp}[1] \subset \mathcal{D}$.*

Lemma 1.3 ([13, §1.3]). *The heart of a bounded t-structure is an abelian category and it determines the bounded t-structure as the extension-closed subcategory generated by the subcategories $\mathcal{A}[j]$ for integers $j \geq 0$.*

In the rest of the text, we will therefore use interchangely the notion of a bounded t-structure or its heart. While it is clear that given a bounded t-structure $\mathcal{L}$, then $\mathcal{L}[n]$ is another bounded t-structure for any integer n, we easily find t-structures that are not the shift of one another. A typical example is provided by the bounded derived category of the representation of the $(n+1)$-th Beilinson's quiver B_{n+1}

$$B_{n+1} \quad = \quad \bullet_1 \overset{\vdots_{n+1}}{\rightrightarrows} \bullet_1 \;\cdots\; \bullet_n \overset{\vdots_{n+1}}{\rightrightarrows} \bullet_{n+1}$$

which has $n+1$ vertices and $n+1$ arrows between any two consecutive vertices:

$$\mathrm{rep}(B_{n+1}) \subset \mathcal{D}^b(\mathrm{rep}(B_{n+1})) \simeq \mathcal{D}^b(\mathbb{P}^n) \supset \mathrm{Coh}\,\mathbb{P}^n.$$

Indeed, any abelian category $\mathcal{A}$ is the heart of a bounded t-structure in its bounded derived category $\mathcal{D}^b(\mathcal{A})$. In the example, $\mathrm{rep}(B_{n+1})$ is a finite heart, while the abelain category $\mathrm{Coh}(\mathbb{P}^n)$ of coherent sheaves on the complex projective space $\mathbb{P}^n$ is not.

A way of producing new t-structures, which are not necessarily standard in the sense above, is via tilting at a torsion pair.

Definition 1.4. *A* torsion pair *in an abelian category $\mathcal{H}$ is a pair of subcategories $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{H} = \mathcal{T} \perp \mathcal{F}$. We call $\mathcal{T}$ the torsion class and $\mathcal{F}$ the torsion-free class.*

It is easy to see that the non-empty intersection of a heart with the aile of another bounded t-structure defines a torsion pair. On the other hand, a torsion pair in the heart of a bounded t-structure $\mathcal{H}$ in a $\mathcal{D}$ defines new bounded t-structures with hearts

$$\mu^\sharp_{\mathcal{F}}\mathcal{H} := \mathcal{T} \perp_{\mathcal{D}} \mathcal{F}[1], \quad \mu^\flat_{\mathcal{T}}\mathcal{H} := \mathcal{F} \perp_{\mathcal{D}} \mathcal{T}[-1]$$

They are called respectively the *forward tilt* at $\mathcal{F}$ and the *backward tilt* at $\mathcal{T}$, and are related by $\mu^\sharp_{\mathcal{T}[-1]}\mu^\flat_{\mathcal{T}}\mathcal{H} = \mathcal{H}$ and $\mu^\flat_{\mathcal{F}[-1]}\mu^\sharp_{\mathcal{F}}\mathcal{H} = \mathcal{H}$, [29]. When we tilt at a torsion(-free) class $\langle S \rangle$ generated by a simple object $S \in \mathcal{H}$, we speak about a simple tilt and we simplify the notation to

$$\mu^\sharp_S\mathcal{H} \quad \text{and} \quad \mu^\flat_S\mathcal{H}.$$

Definition 1.5. *The* exchange graph $\mathrm{EG}(\mathcal{D})$ *of a triangulated category $\mathcal{D}$ is the graph whose vertices are finite hearts of bounded t-structures on $\mathcal{D}$ and whose arrows are either forward or backward simple tilts.*

For the pourposes of these notes, we will consider forward tilts, though clearly this only affect the direction of the arrows.

The group $\mathrm{Aut}(\mathcal{D})$ acts on $\mathrm{EG}(\mathcal{D})$. Since any autoequivalence commutes with the shift functor, if $\Phi \in \mathrm{Aut}(\mathcal{D})$ and $\mathcal{A}$ is the heart of a bounded t-structure, then

$$\Phi\left(\mu^{\flat/\sharp}_{\mathcal{T}/\mathcal{F}}(\mathcal{A})\right) = \mu^{\flat/\sharp}_{\Phi(\mathcal{T}/\mathcal{F})}\Phi(\mathcal{A}). \tag{1.2.3}$$

The following lemma characterises bounded t-structures of a triangulated category. The proof can be deduced from [13, §1.3] and is sketched below.

Lemma 1.6 ([14, Lemma 3.2]). *Let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory of a triangulated category $\mathcal{D}$. Then $\mathcal{A}$ is the heart of a bounded t-structure $\mathcal{L} \subset \mathcal{D}$ if and only if the following two conditions hold:*

(a) *if $k_1 > k_2$ integers, A and B objects of $\mathcal{A}$ then* $\mathrm{Hom}_{\mathcal{D}}(A[k_1], B[k_2]) = 0$*,*

(b) *for every nonzero object $E \in \mathcal{D}$ there is a finite sequence of integers*

$$k_1 > k_2 > \cdots > k_n$$

and a collection of triangles

$$\begin{array}{ccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 \longrightarrow \dots \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\ & \nwarrow & \swarrow \quad \nwarrow & & \swarrow & & \nwarrow & \swarrow & \\ & & A_1 & & A_2 & & & A_n & \end{array}$$

with $A_j \in \mathcal{A}[k_j]$ for all j.

The object A_j appearing in (1.6) are called the k_j-th *cohomology* class of E *with respect to the bounded t-structure*. They are unique up to isomorphism, [14].

Proof. For one direction, we consider $\mathcal{L} = \langle \mathcal{A}[i],\ i \geq 0\rangle_{\mathcal{D}}\rangle$ and $\mathcal{G} = \langle \mathcal{A}[-i],\ i \geq 1\rangle_{\mathcal{D}}\rangle$. By conditions (a) and (b), $\mathcal{G} = \mathcal{L}^{\perp}$. Let $E \in \mathcal{D}$ and $m \in \mathbb{Z}$ be the maximal integer for which $k_i \geq 0$ in (b). Then the cone of the non-zero composite functor $E_m \to E$ lies in $\langle \mathcal{A}[j], k_n \leq j \leq k_{m+1}\rangle \subset \mathcal{G}$, and we have a decomposition $E_m \to E \to G \to E_m[1]$ with $G \in \mathcal{L}^{\perp}$.

The other implication can be proven by using the truncation functors $\tau_{\geq 0}, \tau_{\leq 0}$ and their shifts $\tau_{\geq k}$, $\tau_{\leq k}$, which are defined in [13, §1.3]. See in particular Propositions 1.3.3-1.3.5. Theorem 1.3.6 in loc. cit. shows moreover that the functor $H^k := \tau_{\geq k}\tau_{\leq k} : \mathcal{D} \to \mathcal{A}$ is a cohomological functor. It associates $E \in \mathcal{D}$ with the shifted subfactor $A[-k] \in \mathcal{A}$ appearing in (b).

Last, condition (1.2.2) is equivalent to finiteness of the sequence of triangles appearing in (b). ■

Corollary 1.7. *If $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, then there is an isomorphism of Grothendieck groups*

$$K(\mathcal{A}) \simeq K(\mathcal{D}).$$

Proof. Short exact sequences in $\mathcal{A}$ are precisely the distinguished triangles in $\mathcal{D}$ with three vertices in $\mathcal{A}$. The map $K(\mathcal{A}) \to K(\mathcal{D})$ is induced by the inclusion $\mathcal{A} \subset \mathcal{D}$, while its inverse sends $[E]_{\mathcal{D}}$ to the alternate (finite) sum $\sum_{i\in\mathbb{Z}}(-1)^{k_i}\left[A_i[-k_i]\right]_{\mathcal{A}}$, for A_i and k_i defined in Lemma 1.6, (b). ■

Definition 1.8 ([14, Definition 3.3]). *A* slicing *on the triangulated category $\mathcal{D}$ is a family of full additive subcategories $\mathcal{P} := \{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}} \subset \mathcal{D}$ such that*

(a) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$, *for all $\phi \in \mathbb{R}$,*

(b) *if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$, $j = 1, 2$, then* $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$,

(c) *for any non-zero object $E \in \mathcal{D}$ ther is a finite sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_m$ and a collection of distinguished triangles*

$$\begin{array}{ccccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \to \dots \to & E_{m-1} & \longrightarrow & E_m = E \\ & \nwarrow \quad \swarrow & & \nwarrow \quad \swarrow & & & & \nwarrow \quad \swarrow & \\ & A_1 & & A_2 & & & & A_m & \end{array}$$

with $A_j \in \mathcal{P}(\phi_j)$ for all $j = 1, \dots, m$.

It is not required by definition that $\mathcal{P}(\phi) \neq \{0\}$ for all $\phi \in \mathbb{R}$, nor the non-trivial slices to be dense in $\mathbb{R}$.

As in Lemma 1.6, the decomposition of axiom (c) is unique up to isomorphism, hence one can define $\phi^+_{\mathcal{P}}(E) = \phi_1$ and $\phi^-_{\mathcal{P}}(E) = \phi_n$. For any interval $I \subset \mathbb{R}$, Bridgeland defines

$$\mathcal{P}(I) := \langle \mathcal{P}(\phi) \mid \phi \in I \rangle_{\mathcal{D}}.$$

If I is connected, $\mathcal{P}(I)$ coincides with the subcategory $\langle E \in \mathcal{D} \mid \phi^{\pm}_{\mathcal{P}}(E) \in I \rangle_{\mathcal{D}}$.

Lemma 1.9. *Suppose $I = (0, 1]$ and $\lambda \in I$. Then*

(1) *$\mathcal{P}(I)$ is the heart of a bounded t-structure on $\mathcal{D}$, and*

(2) *$\mathcal{P}((\lambda, 1]) \perp \mathcal{P}((0, \lambda])$ is a torsion pair in $\mathcal{P}(I)$.*

Proof. Both statements follow from the definitions and the conditions of Lemma 1.6, using a truncation functor. ■

The result actually holds for any connected interval of length 1. In particular $\mathcal{P}(I)$ is abelian if I has length 1. It is quasi-abelian if I has length less than 1, [14]. We say that $\mathcal{P}((0, 1])$ is the *heart* of the slicing $\mathcal{P}$.

1.2.2 Bridgeland stability conditions

The notion of a stability condition on a triangulated category was introduced in [14]. The definitions given below (1.11 and 1.12) are the mostly used currently, see also the series of papers by Bayer, Macrì, Stellari, and co-authors. The equivalence of the two definitions is sketched in Theorem 1.15, [14, Proposition 5.3]. The differences with the original definition involve the support condition and the possible dependence on a finite rank lattice.

We start with the preliminary definition of a stablity function on an abelian cateogy and of the Harder-Narasimhan condition.

Definition 1.10. *Let $\mathcal{A}$ be an abelian category and $Z \in \mathrm{Hom}(K(\mathcal{A}), \mathbb{C})$ such that for any $0 \neq A \in \mathcal{A}$,*

$$Z([A]) \in \overline{\mathbb{H}} := \{re^{\pi i \theta} \in \mathbb{R} \mid r \in \mathbb{R}_{>0},\ 0 < \theta \leq 1\}.$$

We say $\frac{1}{\pi}\arg Z([A])$ *the* phase *of* A.

(1) *An object* $A \in \mathcal{A}$ *is said to be* Z-semistable *if, for any non-zero proper sub-object* $B \hookrightarrow A$*, then* $\frac{1}{\pi}\arg Z([B]) \leq \frac{1}{\pi}\arg Z([A])$. *It is called* Z-stable *if the inequality holds strictly.*

(2) Z *is said to be a* stability function *if it satisfies the* Harder-Narasimhan property*: for any* $0 \neq A \in \mathcal{A}$*, there is a finite chain of sub-objects*

$$0 \simeq A_0 \subset A_1 \subset \cdots \subset A_m = A$$

whose quotients $F_j = A_j/A_{j-1}$ *are* Z*-semistable of decreasing phases.*

Note that the notion of semi-stability with respect to a stability function is constant on iso-classes in the Grothendieck group.

Let $\mathcal{D}$ be a triangulated category. We fix a finite rank free lattice $(\Lambda, \langle -, - \rangle)$ together with a surjective group homomorphism $v : K(\mathcal{D}) \twoheadrightarrow \Lambda$. If $\mathcal{D}$ is Hom −finite and $K(\mathcal{D}) \simeq \mathbb{Z}^{\oplus n}$, we take $\left(K(\mathcal{D}), \chi(-,-)\right) \overset{id}{=} \left(\Lambda, \langle -, - \rangle\right)$. In many cases, if $K(|calD)$ has not finite rank, it is standard to consider central charges that factor through the the numerical part, i.e., the quotient of $K(\mathcal{D})$ by the null space of the Euler form on $\mathcal{D}$, or, for some $\mathcal{D} = \mathcal{D}^b(\mathrm{Coh}(X))$, the singular cohomology $H^*(X, \mathbb{Z})$. In the next definition we use the isomorphism of Grothendieck groups of Corollary 1.7.

Definition 1.11. *A* stability condition σ *on* $\mathcal{D}$*,* supported *on the heart* $\mathcal{A}$*, is a pair*

$$\sigma = (\mathcal{A}, Z),$$

consisting on the heart of a bounded t-structure $\mathcal{A}$ *on* $\mathcal{D}$*, together with a* stability function Z *on* $\mathcal{A}$ *that factors through* Λ

$$Z : K(\mathcal{A}) \overset{v}{\twoheadrightarrow} \Lambda \to \mathbb{C},$$

satisfying the support property*: there exists a norm* $||\cdot||$ *on* $\Lambda \otimes \mathbb{R}$ *and a constant* $c \in \mathbb{R}_{>0}$ *such that for any* Z*-semistable* $0 \neq A \in \mathcal{A}$, $|Z(A)| \geq c||v[A]||$.

The homomorphism Z *is referred to as the* central charge.

Note that, since $\Lambda \otimes \mathbb{R}$ is a finite dimensional vector space, all norms are equivalent, hence any definition depending on definition 1.11 will not depend on the choice of the norm.

While the support property looks at a first glance somehow arbitrary and with a different flavour compared with the resto of the definition, it ensures tha the stability function does not take arbitrarily small values, and is crucial in order to define a topology on the set of all stability conditions. It was introduced by M. Kontsevich and Y. Soibelman in [39], where the authors also show that it can be equivalently expressed as the the condition for which there exists a quadratic form $Q : \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ such that

- the kernel of Z is negative definite with respect to Q, and
- $Q(v[A]) \geq 0$ for any Z-semistable object A.

Indeed. if one writes $||\alpha|| = \alpha \cdot \alpha$, then we can define $Q(\alpha, \beta) = \sqrt{Z(\alpha)\overline{Z}(\beta)} - \alpha \cdot \beta$. On the other hand, given $Q(\alpha, \alpha)$, one easily sees that $||\alpha|| = |Z(\alpha)| - Q(\alpha)$ is a norm on $K(\mathcal{A})$. This is most useful when it boils down to a Bogomolov-Gieseker type inequality (e.g., for $\mathcal{D} =$).

A Bridgeland stability condition on a triangulated category is also defined in terms of a slicing parametrizing "distinguished" objects: the semistable ones. Note that here, as well as in Definition 1.11, we drop from the notation the dependence of σ on the choice of (Λ, v).

Definition 1.12. *Let* $K(\mathcal{D}) \overset{v}{\twoheadrightarrow} \Lambda$ *as above. A* stability condition *on* $\mathcal{D}$ *is a pair*

$$\sigma = (Z, \mathcal{P}),$$

where $\mathcal{P}$ *is a slicing on* $\mathcal{D}$ *and* $Z \in \mathrm{Hom}\,(K(\mathcal{D}), \mathbb{C})$ *is a group homomorphism that factors through* $K(\mathcal{D}) \overset{v}{\longrightarrow} \Lambda$ *and satisfies the support property and the following* compatibility *condition: if* $0 \neq E \in \mathcal{P}(\phi)$ *then there exists* $m(E) \in \mathbb{R}_{>0}$ *such that* $Z([E]) = m(E)\exp(i\pi\phi)$.

The following definition is well-posed thanks to Lemma 1.14.

Definition 1.13. *The non-zero objects* $0 \neq E \in \mathcal{P}(\phi)$ *are said to be* σ-semistable of phase ϕ, *and the simples in* $\mathcal{P}(\phi)$ *are said to be* σ-stable.

Lemma 1.14 ([14, Lemma 5.2]). *If a slicing is compatible with a central charge, then any* $\mathcal{P}(\phi)$ *is an abelian category of finite length, i.e., any object has at most finitely many subobjects, up to isomorphism.*

Proof. One can prove that $\mathcal{P}(\phi)$ is abelian by showing that it is closed under kernels and cokernels inside the abelian category $\mathcal{P}((\phi - 1, \phi])$. First show by contraddiction that if $E \to F \to G \to E[1]$ is a distinguished triangle in $\mathcal{P}((\phi-1, \phi])$, then $\phi^+(E) \leq \phi^+(F)$ and $\phi^-(F) \leq \phi^-(G)$. Then use the compatibility condition.
The finite length property is ensured by the support property of the central charge. ■

Theorem 1.15. *Definition 1.11 and Definition 1.12 are equivalent. The* σ*-(semi)stable objects of* $\mathcal{D}$ *are exactly the* Z*-(semi)stable objects of* $\mathcal{P}((0, 1])$ *and all their shifts.*

Sketch of the proof. If we have a stability condition $\sigma = (Z, \mathcal{A})$ in the sense of Definition 1.11, then the collection $\mathcal{P} := \{\mathcal{P}(\phi), \phi \in \mathbb{R}\}$ defined by

$$\mathcal{P}(\phi) := \{E \in \mathcal{A},\ Z\text{-semistable in } \mathcal{A} \text{ of phase } \phi\} \cup \{zeroes\}, \quad 0 < \phi \leq 1$$

and

$$\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1],$$

is a slicing on $\mathcal{D}$, compatible with Z regarded as a group homomorphism on $K(\mathcal{D})$.

On the other hand, a slicing $\mathcal{P}$ defines a bounded t-structure $\mathcal{D}_{>0} := \mathcal{P}((0, +\infty))$ on $\mathcal{D}$ with heart $\mathcal{P}((0, 1])$, and $Z \in \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ induces a stability function on $\mathcal{P}((0, 1])$. ∎

From now on, with slight abuse of notation, we will write $Z(E)$ for $Z([E])$. For any non-zero object E in $\mathcal{D}$ one defines its *mass* with respect to a chosen stability condition as

$$m_\sigma(E) = \sum_j |Z(A_j)| \in \mathbb{R}_{>0},$$

where the A_j are the Harder-Narasimhan factors with respect to σ, i.e., the objects, unique up to isomorphism, appearing in Definition 1.8, c) for the underlying slicing. Note that the mass of an object cannot vanish, but the central charge can vanish. If an object $X \in \mathcal{D}$ is σ-semistable and belongs to $\mathcal{P}(\phi)$, then $\phi^+(X) = \phi^-(X) = \phi = \frac{1}{\pi} \arg Z(X)$, and $m_\sigma(X) = |Z(X)|$. This is true for all the simple objects of the supporting heart.

The set of all Bridgeland stability conditions on a triangulated category, for a fixed choice of (Λ, ν) is denoted $\mathrm{Stab}_{(\Lambda,\nu)}(\mathcal{D})$. Even when a stability condition on a given triangulated category is known to exist, computing the whole $\mathrm{Stab}_{(\Lambda,\nu)}(\mathcal{D})$ can be very hard.

1.2.3 Stability functions

The notion of Bridgeland stability conditions on a triangulated category was inspired by the work of Douglas [2, 24] on Π-stability for D-branes, and, more in general, by ideas from string theory. These ideas have driven part of the mathematical research on the stability manifold since its definition.

On the other hand, Bridgeland stability provides the first example of stability conditions on a triangulated category, and chosing a central charge on a heart appears like a natural generalization of previously known notions of stability conditions on an abelian category, their key property being the Harder-Narasimhan property. The tyical example is slope stability, but it is not always true that stability in abelian sense can be promoted to Bridgeland stability.

Slope stability. We consider slope stability defined by Alastair King for abelian category of representations of quivers and module categories. Let us take Q an acyclic finite quiver, and $\underline{V} = (V_i, f_\alpha)_{i,\alpha}$ a representation of Q. Fix $\underline{a} \in \mathbb{Z}^{|Q^0|}$ such that $\sum_i a_i d_i = 0$ for some dimension vector $\underline{d}$, and set

$$\mu_{\underline{a}}(\underline{V}) = \frac{\sum_i a_i \dim V_i}{\sum_i \dim V_i}.$$

We say that a representation $\underline{V}$ is $\mu_{\underline{a}}$-semistable if $\mu_{\underline{a}}(\underline{V}) = 0$ and, for any sub-representation $\underline{W} \subseteq \underline{V}$, then $\mu_{\underline{a}}(\underline{W}) \geq 0$. It is called $\mu_{\underline{a}}$-stable if the only sub-representations with $\mu_{\underline{a}}(\underline{W}) = 0$ are the trivial ones. The key result by King, [38], concernes the existence of moduli spaces of $\mu_{\underline{a}}$-semistable Q-representations of fixed dimension $\underline{d}$ as a projective variety. It is done using GIT techniques.

In general a slope function on $\mathrm{rep}(Q)$ is given by two additive functions $c : \mathrm{rep}(Q) \to \mathbb{R}$ and $r : \mathrm{rep}(Q) \to \mathbb{R}_{>0}$ as $\mu(\underline{V}) = \frac{c(\underline{V})}{r(\underline{V})}$, and mimic the analogous notion by Mumford for vector bundles (where c and r are the degree, depending on the choice of a polarization on X, and the rank, respectively), extended to the abelian category of coherent sheaves over a curve X.

The slope functions satisfies the Harder-Narasimahn property, i.e., for any $\underline{V} \in \mathrm{rep}\, Q$ there exist $\underline{F}^0 = \underline{V} \supset \underline{F} \supset \cdots \supset \underline{F}^k = 0$ such that

$$\underline{F}^j / \underline{F}^{j+1}$$

are semistable of decreasing slope. Positivity and finiteness properties allow us to regard at a slope function on $\mathrm{rep}(Q)$ as a stability function in the sense of Definition 1.10 by setting $Z_\mu(\underline{V}) = -c(\underline{V}) + ir(\underline{V})$. Similarly for coherent sheaves of pure dimension on a curve X, taking $\Lambda = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$. Note, however, that, if X is not a curve, this argument doesn't work.

A systematic study of stability functions and the Harder-Narasimhan property in the abelian context was carried out by Rudakov. See for example [54] and subsequent papers.

Finite hearts. A special case is that of finite abelian categories. Suppose $\mathcal{D}$ has finite rank Grothendieck group $K(\mathcal{D}) \simeq \mathbb{Z}^{\oplus n}$ and a bounded t-structure with a finite, finite length heart $\mathcal{H}$. Let $\mathrm{Sim}(\mathcal{H}) = \{S_1, \ldots, S_n\}$ be the set of simple objects of $\mathcal{H}$. Than any group homomorphism $Z \in \mathrm{Hom}(K(\mathcal{H}), \mathbb{C})$, such that $Z(S_i) \in \mathbb{H}$, automatically satisfies the Harder-Narasimhan condition and the support property, and therefore is a stability function on the heart.

1.3 The stability manifold

Let $\mathcal{D}$ be a Hom-finite triangulated category, $\nu : K(\mathcal{D}) \twoheadrightarrow \Lambda$ as in Section 1.2. The set of Bridgeland stability conditions on $\mathcal{D}$ factoring through $K(\mathcal{D}) \xrightarrow{\nu} \Lambda$ here is denoted

$$\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}_{(\Lambda,\nu)}(\mathcal{D}).$$

We remove the dependence on (Λ, ν) also from the notation for a single stability condition.

The main result in [14] is that $\mathrm{Stab}(\mathcal{D})$ can be given the structure of a complex manifold. The goal of this section is to review the complex structure and the main properties of the stability manifold, provide few well-known examples, and introduce some old and new questions. For simplicity, and abusing notation, we use the expression central charge both for the map $Z : K(\mathcal{D}) \to \mathbb{C}$ and for the induced map $\Lambda \to \mathbb{C}$.

1.3.1 The complex structure

The map $d : \mathrm{Stab}(\mathcal{D}) \times \mathrm{Stab}(\mathcal{D}) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ defined by

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi^-_{\sigma_2}(E) - \phi^-_{\sigma_1}(E)|, |\phi^+_{\sigma_2}(E) - \phi^+_{\sigma_1}(E)|, \left| \log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \right| \right\} \tag{1.3.1}$$

is a generalized metric on $\mathrm{Stab}(\mathcal{D})$, i.e., it satisfies the axiom of a metric space except that it need not be finite, [14]. We will loosely refer to it as a metric. As a consequence, it defines a topology on $\mathrm{Stab}(\mathcal{D})$ and induces a metric space structure on each connected component. We consider $\mathrm{Stab}(\mathcal{D})$ as endowed with the metric topology. Equivalently, the topology is induced by the generalized metric

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi^-_{\sigma_2}(E) - \phi^-_{\sigma_1}(E)|, |\phi^+_{\sigma_2}(E) - \phi^+_{\sigma_1}(E)|, ||Z_1 - Z_2||_{\Lambda^*_{\mathbb{C}}} \right\},$$

where $||W||_{\Lambda^*_{\mathbb{C}}}$ denotes the operator norm on $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$. It is easy to relate the use of the operator norm here with the support property of Definition 1.11, that can be rewritten as

$$\inf \left\{ \frac{|Z(E)|}{||v[E]||_{\Lambda_{\mathbb{R}}}} : 0 \neq E \text{ semistable} \right\} > 0.$$

According to the metric d defined above, the distance between two stability conditions depends both on how "different" the central charges are, and how further apart the hearts of the slicings are. For instance, if two stability conditions $\sigma_1 = (Z, \mathcal{A})$ and $\sigma_2 = (Z, \mathcal{A}[2n])$ differ by the choice of shifted hearts of bounded t-structures, then their distance is $2n$. Allowing arbitrarily small central charges, that would possibly lead to infinitely distant stability conditions, is prevented by definition, and on each connected component the generalized metric defined in (1.3.1) is finite and complete (see [10, 58] for details). Some metric properties of the stability manifold have been studied by Woolf, [58].

Theorem 1.16 ([14, Theorem 1.2], [10]). *When not empty, the space* $\mathrm{Stab}(\mathcal{D})$ *is a complex manifold of dimension* $\mathrm{rank}\,\Lambda$*, locally isomorphic to* $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$ *via the forgetful morphism*

$$\mathcal{Z} : \sigma = (Z, \mathcal{P}) \mapsto Z. \tag{1.3.2}$$

Theorem 1.16 means that it is enought to deform the central charge in order to cover any small neighbourhood of a stability condition in $\mathrm{Stab}(\mathcal{D})$. In fact, its proof is based on the deformation properties of the central charge, proved in [14, §7], that, in turn, are guaranteed by the support property. Some remarks are due. The original request, for the space to be well-behaved, was denoted "locally-finiteness" (Definition 5.7 in [14]). It is implied by the support property appearing in the currently accepted definitions, see [10, 39]. The local homeomorphism $\mathcal{Z}$ of (1.3.2) showed in [14] was promoted to a local isomorphism in [10, Appendix A]. An alternative proof of Theorem 1.16 is given in the reccommended paper [8] by A. Bayer.

We restrict to a connected component of the space $\mathrm{Stab}(\mathcal{D})$.

Lemma 1.17 ([58, Corollary 5.2]). *If $\sigma_1 = (\mathcal{A}_1, Z_1)$ and $\sigma_2 = (\mathcal{A}_2, \mathbb{Z}_2)$ are in the same connected component of $\mathrm{Stab}(\mathcal{D})$, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are related by a finite sequence of forward or backward tilts at some (possibly trivial) torsion pairs.*

This lemma becomes more concrete under some finiteness assumption on $\mathcal{D}$. Let $\mathcal{H}$ be the heart of a bounded t-structure on $\mathcal{D}$. We denote by $\mathrm{Stab}(\mathcal{H}) \subset \mathrm{Stab}(\mathcal{D})$ the subset consisting on stability conditions supported on $\mathcal{H}$. Subsets $\mathrm{Stab}(\mathcal{H})$, as $\mathcal{H}$ varies, partition $\mathrm{Stab}(\mathcal{D})$. They need not be either open or closed. As remarked in Subsection 1.2.3, if $\mathcal{H}$ is a finite length heart, with finitely many simples $\mathrm{Sim}(\mathcal{H}) = \{S_1, \ldots, S_n\}$, then any group homomorphism $Z \in \mathrm{Hom}(K(\mathcal{H}), \mathbb{C})$, such that $Z(S_i) \in \mathbb{H}$, automatically satisfies the Harder-Narasimhan condition and the support property. The following Lemma is therefore clear and we describe how to "glue" such pieces in Proposition 1.19.

Lemma 1.18. *If $\mathcal{H}$ is a finite length heart with $n < \infty$ simples, then $\mathrm{Stab}(\mathcal{H}) \subset \mathrm{Stab}(\mathcal{D})$ is isomorphic to $\mathbb{H}^n$.*

Proposition 1.19 ([15, Section 5], [57, Proposition 2.6, Corollary 2.10]). *Let $\mathcal{A}_1$ be the heart of a bounded t-structure in $\mathcal{D}$ and suppose that $\mathcal{A}_1$ has finite lenght and finitely many simples. Let S be a simple object in $\mathcal{A}_1$. If $\emptyset \neq \mathcal{W}_S \subset \mathrm{Stab}(\mathcal{H})$ is the real-codimension 1 locus for which a S has phase 1, and all other simples have phase in $(0, 1)$, we have*

$$\mathrm{Stab}(\mathcal{A}_1) \cap \overline{\mathrm{Stab}(\mathcal{A}_2)} = \mathcal{W}_S \Longleftrightarrow \mathcal{A}_2 = \mu_S^\flat \mathcal{A}_1.$$

If $\mathcal{W}$ is the subset of $\mathrm{Stab}(\mathcal{A}_1)$ of stability conditions such that k simples $S_{i_1}, \ldots, S_{i_k}$ have phase 1 and the others have phase less than 1, we have

$$\mathcal{W} \subseteq \mathrm{Stab}(\mathcal{A}_1) \cap \overline{\mathrm{Stab}(\mathcal{A}_2)} \Longleftrightarrow \mathcal{A}_2 = \mu_\mathcal{T}^\flat \mathcal{A}_1$$

for some torsion class $\mathcal{T} \subset \langle S_1, \ldots, S_k \rangle_{\mathcal{A}_1}$. The real dimension $\dim_\mathbb{R} \mathrm{Stab}(\mathcal{A}_1) \cap \overline{\mathrm{Stab}(\mathcal{A}_2)}$ is at least k.

Proof. The proof is based on the $\mathbb{C}$-action defined below. The inclusion of $\mathcal{W}$ need not be an equality. ∎

The real-codimension 1 boundaries $\mathcal{W}_S$ of sets $\mathrm{Stab}(\mathcal{A})$ are sometimes called *walls (of second type)*. The connected components of the complement of the closure of the union of walls in $\mathrm{Stab}(\mathcal{D})$ are often called a *chambers*.

Another type of *wall and chamber* decomposition of the stability manifold is given by so-called walls of *marginal stability*. They are the set $\mathcal{W}_\alpha(\beta)$ where the central charge of non-proportional classes $\alpha, \beta \in K(\mathcal{D})$ with non-trivial extension, satisfies $Z(\alpha)/Z(\beta) \in \mathbb{R}$. Along these walls, phenomena of strictly semistability may happen, and in fact they may identify regions of the stability manifold on which the property of being stable of an object changes.

1.3.2 Group actions

A question is whether we can cover a whole connected component of the stability space $\mathrm{Stab}(\mathcal{D})$ by starting at some known family of stability conditions and acting by a group. While in general this is not true, there are several examples where this strategy allows to compute $\mathrm{Stab}(\mathcal{D})$ or an appropriate quotient, usually $\mathrm{Stab}(\mathcal{D})/G$ for a subgroup $G \subset \mathrm{Aut}(\mathcal{D})$.

There are two natural actions on $\mathrm{Stab}(\mathcal{D})$ induced respectively by autoequivalences of the category and by the orientation-preserving transformation of $\mathbb{C}$. None of these actions change the set of semistable objects, and the result on the slicing is essentially a relabelling.

The group of autoequivalences $\mathrm{Aut}(\mathcal{D})$ acts on the left on $\mathrm{Stab}(\mathcal{D})$ by isometries

$$\Phi.(\mathcal{H}, Z) = \left(\Phi(\mathcal{H}), Z \circ [\Phi]^{-1}\right),$$

or, equivalently,

$$\Phi.(Z, \mathcal{P}) = \left(Z \circ [\Phi]^{-1}, \{\Phi(\mathcal{P}(\phi))\}_{\phi \in \mathbb{R}}\right).$$

Here $[\Phi]$ denotes the map induced by $\Phi \in \mathrm{Aut}(\mathcal{D})$ on the Grothendieck group $K(\mathcal{D})$. Note that there is a special subgroup $\mathrm{Aut}^0(\mathcal{D}) \subset \mathrm{Aut}(\mathcal{D})$ consisting on auto-equivalences that induce the identity on the Grothendieck group. The n-th shift functor $[n] \in \mathrm{Aut}(\mathcal{D})$ acts by

$$[n].(\mathcal{H}, Z) = (\mathcal{H}[n], (-1)^n Z).$$

The universal covering $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ of the group $\mathrm{GL}^+(2,\mathbb{R})$ of 2×2 matrices with real entries and positive determinant, acts smoothly on the right as follows. We identify

$$\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) = \left\{ (T, f) \mid \begin{array}{l} T : \mathbb{R}^2 \to \mathbb{R}^2 \in \mathrm{GL}^+(2,\mathbb{R}) \\ f : \mathbb{R} \to \mathbb{R} \text{ increasing, } f(\phi + 1) = f(\phi) + 1 \\ f_{|\mathbb{R}/2\mathbb{Z}} \equiv T_{|\mathbb{R}^2/\mathbb{R}_{>0}} \end{array} \right\},$$

and define the image of $\sigma = (Z, \mathcal{P})$ under (T, f) as

$$\left(T^{-1} \circ Z, \{\mathcal{P}(f(\phi))\}_{\phi \in \mathbb{R}}\right).$$

The $\mathrm{Aut}(\mathcal{D})$-left action and the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-right action commute, and also commute with the action by scalars (that we write here on the left, instead of taking $\mathbb{C} \subset \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$)

$$\lambda.(Z,\mathcal{P}) = (e^{-\pi i \lambda} Z, \mathcal{P}') \quad \text{where } \mathcal{P}'(\phi) = \mathcal{P}(\phi + \operatorname{Re}\lambda), \tag{1.3.3}$$

for $\lambda \in \mathbb{C}$. It coincides with the action of $[n] \in \mathrm{Aut}(\mathcal{D})$ when $\lambda = n \in \mathbb{Z}$. Note that $\mathbb{C}$-orbits $\mathbb{C}.\sigma = \{\lambda.\sigma \mid \lambda \in \mathbb{C}\}$ are closed, and the restriction of the metric d to $\mathbb{C}.\sigma$ is given by

$$d(\sigma, \lambda.\sigma) = \max\{|\operatorname{Re}\lambda|, \pi|\operatorname{Im}\lambda|\}.$$

The real part of λ produces a modification that can be pictorially described as a "rotation" of the heart of the t-structure, because $e^{-\pi i \lambda} Z$ identifies a different distinguished half-plane in the complex plane, or as a translation of the heart of the slicing, as it is affine on the set of phases of semi-stable objects. The imaginary part is responsible for the rescaling of the central charge. If $0 < \operatorname{Re}\lambda \leq 1$, the $\mathbb{C}$-action on a stability condition $\sigma = (\mathcal{A}, Z)$ represented as in Definition 1.11 gives

$$\lambda.(Z,\mathcal{A}) = (e^{-\pi i \lambda}, \mu^{\sharp}_{\mathcal{F}_\lambda}\mathcal{A}),$$
$$(-\lambda).(Z,\mathcal{A}) = (e^{\pi i \lambda}, \mu^{\flat}_{\mathcal{T}_\lambda}\mathcal{A}),$$

where $\mathcal{F}_\lambda$ is the torsion-free class

$$\mathcal{F}_\lambda = \mathcal{P}((0, \operatorname{Re}\lambda]) = \langle E \in \mathcal{A} \mid E \text{ semistable}, \phi_\sigma(E) \leq \operatorname{Re}\lambda \rangle,$$

and $\mathcal{T}_\lambda$ is the torsion class

$$\mathcal{T}_\lambda = \mathcal{P}((1 - \operatorname{Re}\lambda, 1]) = \langle E \in \mathcal{A} \mid E \text{ semistable}, \phi_\sigma(E) > \operatorname{Re}\lambda \rangle.$$

The space $\mathbb{P}\,\mathrm{Stab}(\mathcal{D}) := \mathrm{Stab}(\mathcal{D})/\mathbb{C}$ is called the *projectivized stability space*. It is a complex projective manifold locally modelled on $\mathbb{P}\,\mathrm{Hom}(\Lambda, \mathbb{C})$.

1.3.3 Some questions regarding Stab($\mathcal{D}$)

Despite the attention that Bridgeland stability conditions and the stability manifold have attracted since their introductions, a general strategy for constructing a stability structure on a triangulated category is not known yet. The definition problem appears usually when we deal with geometric categories, such as the bounded derived categories of complex varieties. At the same time, saying that the space $\mathrm{Stab}(\mathcal{D})$ is empty might be even harder.

To my experience, there are two main research directions concerning the stability manifold of a triangulated category from representation theory.

Mirror Symmetry. One arises in the context of Mirror Symmetry and has to do with a theory of invariants counting (in the appropriate sense) semi-stable objects, and with encoding such invariants in some additional geometric structures, which are analogous to other structures appearing in other moduli problems, especially Gromov-Witten theory. These structures involve pencils of isomonodromic connections on the tangent bundle to the space $\mathrm{Stab}(\mathcal{D})$, e.g., [6, 7, 17, 25]. The enumerative theory associated with Bridgeland stability conditions is often called Donaldson-Thomas theory (in analogy with counting of sheaves on a Calabi-Yau three-fold) or a theory of BPS indeces (where this notation is borrowed from physics). While such theories are not completely developed yet, quiver categories provide examples to start with, [47].

Classical questions. The other direction has to do with computing the whole $\mathrm{Stab}(\mathcal{D})$ and studying it as a topological and complex space. While this has a more classical flavour, an interesting feature of this complex space associated with a category $\mathcal{D}$ is that studying its topology and geometry usually requires deep understanding of bounded t-structures on $\mathcal{D}$. Woolf, in the already mentioned paper [57], explains relations between the topology of $Stab(\mathcal{D})$ and tilting, under suitable assumptions.

We usually restrict to a connected component of the space $\mathrm{Stab}^\circ(\mathcal{D})$. We chose it by requiring that it contains stability conditions supported on a fixed chosen heart of a bounded t-structure on $\mathcal{D}$. In general, the space $\mathrm{Stab}(\mathcal{D})$ might not be connected (as shown in examples in [46] and in [4]), thought it is often conjectured to be connected. It is proven in some cases, e.g., for $\mathcal{D}^b\left(\mathrm{rep}(\bullet \overset{n}{\Rightarrow} \bullet)\right)$ (including the derived category of $\mathbb{P}^1$) in [43], and for the derived categories of coherent sheaves on the minimal resolutions of A_n-singularities supported at exceptional sets, which also admit a description in quivers terms, in [31], to name some early examples.

Using homological tools and the study of bounded t-structures, there are results about simply-connectness and contractibility of connected components of stability manifolds. A very incomplete list of examples in this direction inlcudes [1, 49, 52].

On the other hand, any non-empty connected component $\mathrm{Stab}^\circ(\mathcal{D})$ of its stability manifold is non-compact. (Partial) compactifications of (a connected component of) the stability manifold or a quotient by the groups $\mathbb{C}$ or $\mathrm{Aut}(\mathcal{D})$ have recently been proposed and studied in few classes of examples. In [3] the authors consider the closure of the image of an embedding of $\mathrm{Stab}^\circ(\mathcal{D})/\mathbb{C}$ in a projective space, and define a *Thurston compactification*. In [22] and [5], infinitesimal deformations of the mass function or the central charge are introduced in such a way that they induce stability conditions on appropriate triangulated quotients of $\mathcal{D}$. The two strategies lead to the notion of *Lax stability conditions* and *multi-scale stability conditions*, respectively.

1.3.4 Some examples

We collect an incomplete list of references of computations of stability manifolds, before focusing on one specific example in the next Section.

Stability conditions for geometric categories. The stability manifold of $\mathcal{D}^b(\mathrm{Coh}\, C_g)$, where C_g is a curve of genus $g = 0$ or $g \geq 1$, was computed by Bridgeland and by Macrì at very early stages. They were followed by K3 surfaces (summarized in [44]), and Calabi-Yau three-folds, which are the natural target spaces of corresponding theories in physics and the ambient of conjectures relating invariants from different theories. Perhaps the most investigated Calabi-Yau three-fold was the quintic three-fold $x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \in \mathbb{P}^4\mathbb{C}$, which is an interesting variety from many points of view in Mirror Symmetry. It was completed only in 2018 in [41] after great efforts. Computing the stability manifold for the bounded derived category of a variety X of dimension 3 and higer is complicated, see [9, 11] and references in the introduction of [42]. A strategy to construct a stability condition by Bayer, Macrì, Toda is called tilt-stability, [12]. With this procedure, a weaker notion of stability is constructed on $\mathrm{Coh}\, X$, and deformed to induce a honest stability condition on an appropriate heart.

Stability manifolds for derived and Calabi-Yau quiver categories. When we deal with quiver categories we can count on some amount of combinatorics, and they therefore represent a good starting point for testing conjectures related with Bridgeland theory. On the other hand, the study of their stability manifold has sometimes revealed independently interesting features. Some examples of results concerning the stability manifold for quiver categories are [18, 20, 21, 30, 45, 53].

Stability manifolds related with finite-dimensional complex Lie algebras. Let $\mathfrak{g}_\Gamma$ be the complex Lie algebra associated with the Dynkin quiver Γ. The spaces of stability conditions of certain triangulated categories $\mathcal{D}_\Gamma$ associated with Γ are related with $\mathfrak{g}_\Gamma$ in a way that depends on the category and its autoequivalences. The results involve isomorphisms between (a connected component of) a stability manifold and (the universal cover of) the quotient of a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_\Gamma$ by a Weyl group. Some of these categories, and their stability manifolds, are considered for instance in [16, 18, 32, 56]. Beside this relation being interesting in itself and providing different incarnations of the theory of Dynkin diagrams, it also provides an example of stability manifolds enriched with additional and conjectured geometric structure.

1.4 An example: the stability manifold of CY_3-Ginzburg categories

In this Section we review the description of (a connected component) of the stability manifolds of a class of triangulated categories, defined in 1.4.1 below and denoted $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$, which is known thanks to the Bridgeland-Smith correspondence relating stability conditions and a class of meromorphic quadratic differentials. After the correspondence (Theorem 1.22), we state few consequences, concerning the moduli space of stability conditions on $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ and the moduli space of quadratic differentials, respectively. The last subsection briefly mentions some generalizations of the Bridgeland-Smith correspondence.

The idea behind this Section is to emphasize some tools that might be useful in order to describe the stability manifold, and some fruitful interaction between two apparently very different moduli problems.

1.4.1 CY_3-Ginzburg categories $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$

Ginzburg categories are triangulated categories of Calabi-Yau dimension $N \geq 3$. They are traditionally defined starting form the combinatorial data of a quiver with potential appropriately modified and graded. We refer to Appendix 1.A for background on quivers and the Ginzburg algebra, and for the notion of decorated marked surface and their exchange graphs.

Here we will be interested in a sub-class of Ginzburg categories that have CY dimension 3 and are obtained from a quiver with potential dual to a triangulation of a decorated marked Riemann surface. For this reason, we slighty change the approach and start with a a simply decorated (unpunctured) marked bordered Riemann surface

$$\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}} = (\mathbf{S}, \mathbb{M}, \Delta, \mathbf{w} \equiv \mathbf{1}),$$

of genus g, in the sense of Section 1.A.2. Notice that once g and a partition $\mathbf{m} = (m_i)_{i=1}^{b}$ of the cardinality $|\mathbb{M}|$ of $\mathbb{M}$ are fixed, the compatibility condition 1.A.1 fixes the number of decorations in the interior of a surface $\mathbf{S}$ underlying $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$ and determines $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$. The choice of writing the whole t-uple is aimed to remark that a set of decorations has been fixed, and flips of arcs are relative to the decorations.

We fix an initial triangulation $\mathbb{T}$ of $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$ separating the decorations, and its dual quiver $(Q_{\mathbb{T}}, W_{\mathbb{T}})$. We fix $N = 3$ and consider the Ginzburg algebra $\Gamma_3(Q_{\mathbb{T}}, W_{\mathbb{T}})$, defined in section 1.A.1. We define the triangulated CY_3 category

$$\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}) := \operatorname{pvd}\Gamma_3(Q_{\mathbb{T}}, W_{\mathbb{T}}), \tag{1.4.1}$$

and we call it the Ginzburg category associated with the data $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$. By definition, the category $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ admits a bounded t-structure with finite heart $\mathcal{H} := \operatorname{mod}\mathcal{J}(Q_{\mathbb{T}}, W_{\mathbb{T}})$, that we call *standard* t-structure (and standard heart).

Theorem 1.20. *The category $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ does not depend on the choice of the initial triangulation, but only on the data $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$. It is* Hom*-finite and has Calabi-Yau dimension 3.*

Proof. The first statement follows from a correspondence of flips of triangulations of the un-decorated underlying surface and mutations of their dual quivers, and the equivalence $\mathrm{pvd}(Q, W) \simeq \mathrm{pvd}(Q', W')$ whenever (Q, W) and (Q', W') are related by mutation, due to Keller-Yang [35]. The full statement, promoted to the decorated setup, follows from [36]. The second statement is proven in [35] in the more general context of perfectly valued categories over a Ginzburg algebra. ■

We consider the full exchange graph $\mathrm{EG}(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$ and we restrict to the connected component containing the vertex corresponding to the standard heart $\mathcal{H}$. We denote it by $\mathrm{EG}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$. Let $\mathrm{sph}(\mathcal{H})$ be the spherical twist group of $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$, i.e., the subgroup of $\mathrm{Aut}\,\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ generated by the set of twists $\{\Phi_S \mid S \in \mathrm{Sim}\,\mathcal{H}\}$ defined by

$$\Phi_S(X) = \mathrm{Cone}(S \otimes \mathrm{Hom}^\bullet(S, X) \to X).$$

The following Theorem was proven in [36] and builds on previous versions by Labardini-Fragoso.

Theorem 1.21 (King-Qiu). *There are isomorphism of infinite exchange graphs*

$$\mathrm{EG}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})) \simeq \mathrm{Exch}^\circ(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}), \tag{1.4.2}$$

and of finite exchange graphs

$$\mathrm{EG}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))/\mathrm{sph}(\mathcal{H}) \simeq \mathrm{EG}(\mathbf{S}, \mathbb{M}). \tag{1.4.3}$$

Beside stating a bijection between a set $|\mathrm{EG}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))|$ of finite hearts of $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ and triangulations of $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$, Theorem 1.21 proves a correspondence between the operation of simple tilt on bounded t-structures of $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ and forward mutations of arcs (relative to a set of decorations) on $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$.

1.4.2 Stability conditions as quadratic differentials

We fix $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$ as above and an initial triangulation $\mathbb{T}$ whose dual quiver with potential has set of vertices Q_0. We let $\Lambda := \mathbb{Z}^{|Q_0|}$.

In analogy with the notation used for the exchange graph $\mathrm{EG}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$ of $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$, the symbol $^\circ$ in $\mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$ identifies the connected component (principal part) of the stability manifold $\mathrm{Stab}\,\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ containing stability conditions supported on the standard heart. On groups of autoequivalences of the category it identifies the subgroup of those that preserve the principal part. The subscript $_K$ on groups of autoequivalences refers to functors that moreover act as the identity on the Grothendieck group. The groups $\mathcal{A}ut^\circ$ and $\mathcal{A}ut^\circ_K$ are the quotients of $\mathrm{Aut}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$ and $\mathrm{Aut}^\circ_K(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$ by the corresponding subgroups of negligible autoequivalences, i.e., those that act trivially on $\mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$. The forgetful map defined in (1.3.2), restricted to $\mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$, is appropriately composed with the isomorphism $\Lambda \simeq K(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$ and still denoted $\mathcal{Z}$.

In the next Theorem, $\mathrm{Quad}_g(1^r, \mathbf{m})$ denotes the space of meromorphic quadratic differentials on a Riemann surface of genus g with simple zeroes and poles of order $m_i + 2$. Quadratic differentials can be framed in several ways: $\mathrm{Quad}_g^{\Lambda,\circ}(1^r, \mathbf{m})$ denotes a connected component of the space of Λ-framed quadratic differentials, while $\mathrm{FQuad}^\circ(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ denotes a connected component of the space of Teichmüller framed quadratic differentials. In both cases the connected component is specified by the choice of a triangulation $\mathbb{T}$ of $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$. These moduli spaces and the period map $\int_*$ involved in Theorem 1.22 are defined in Appendix 1.A.3.

Theorem 1.22 (Bridgelan-Smith correspondence). *There is an isomorphism of complex manifolds that fits into a commutative diagram*

$$\begin{array}{ccc} K : \mathrm{FQuad}^\circ(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}) & \xrightarrow{\quad\simeq\quad} & \mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})) \\ & {\scriptstyle\int_*}\searrow \qquad \swarrow{\scriptstyle\mathcal{Z}} & \\ & \mathrm{Hom}(\Lambda, \mathbb{C}) & \end{array} \tag{1.4.4}$$

and is equivariant with respect to the action of the mapping class group $MCG(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ on the domain and of the group $\mathcal{A}ut^\circ(\mathcal{D})$ on the range. It descends to isomorphisms of complex orbifolds

$$K^\Lambda : \mathrm{Quad}_g^{\Lambda,\circ}(1^r, \mathbf{m}) \to \mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))/\mathcal{A}ut_K^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})) \tag{1.4.5}$$

$$\overline{K} : \mathrm{Quad}_g(1^r, \mathbf{m}) \to \mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))/\mathcal{A}ut^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))\,. \tag{1.4.6}$$

Defining explicitely the isomorphisms is beyond the scope of these notes, and we limit ourselves to a panoramic view. We recommend, however, the interested reader to read the Introduction to [20] for full understanding of the correspondence.

The original Bridgeland-Smith correspondence is about the existence of the map K^Λ and is the content of [20, Theorem 11.2] proved in Section 11 of loc.cit. It was inspired by the work of the physicists Gaiotto, Moore, and Neitzke. In facts a version of equation (1.4.5) holds more widely for Ginzburg categories of Calabi-Yau dimension 3 associated with quivers with potential from a triangulation induced by a quadratic differentials with poles of order $m_j + 2 \geq 0$ and simple zeroes (with few exceptions, listed in [20, Definition 9.3] - see also Section 11.6), at the cost of replacing the space $\mathrm{Quad}_g^{\Lambda,\circ}(1^r, \mathbf{m})$ with an appropriate bigger orbifold described in [20, Section 6]. Here it is not considered to avoid technicalities. The construction of the map K^Λ relies on previous results by Labardini-Fragoso [40] on a correspondence between flips of arcs in the un-decorated surface and mutations of the quiver, and uses the combinatorial description of the Ginzburg category $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ in terms of quivers with potential. This explain why the result is up to to the action of the group $\mathcal{A}ut_K^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$.

The lift K of equation (1.4.4) is constructed in [36, Theorem 4.13], where the combinatorial description of the category $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ defined from a quiver with potential is enhanced to data from an arcs system on a simply decorated marked surface, and the operation of mutation of quivers is promoted to flips of arcs relative to decorations. Last, the quotient $\overline{K}$ is added in [4], where the reader can also find a more technical, but still concise, sketch of the proof of the whole theorem. In fact, the correspondence is stated here as it appears in [4, Theorem 8.1].

In the rest of the subsection we recall some consequences, already emphasized in [20], that are intimately related with the construction of the isomorphisms of Theorem 1.22, and briefly present a simple example.

The exchange graph is a skeleton. The main idea behind the correspondence is that a configuration of open arcs (a triangulation) on $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$ singles out a finite bounded t-structure on $\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ and flipping (isotopy classes of) arcs behaves like simple tilts of hearts, [36,40]. Figure 13 in [4] gives a hint on how a flip is induced by continuously deforming a quadratic differential, or equivalently the position of its zeroes.

A consequence of this correspondence is that the exchange graph $\mathrm{EG}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$, is a “skeleton” for the space $\mathrm{Stab}^\circ(\mathcal{D}_3)$, which is *tame* or *generically finite*.

Corollary 1.23. *The space* $\mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$ *is* tame *or* generically finite*, i.e.,*

$$\mathrm{Stab}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})) = \mathbb{C}\cdot \bigcup_{\mathcal{H}\in\mathrm{EG}^\circ} \mathrm{Stab}\,\mathcal{H},$$

where EG° *stands for* $\mathrm{EG}^\circ(\mathcal{D}_3(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}))$.

In fact, the isomorphisms of Theorem 1.22 are first constructed on the generic locus of meromorphic quadratic differentials with no horizontal saddle connections, that correspond to generic stability conditions supported on a finite heart, that is a heart in the set of vertices $|\,\mathrm{EG}^\circ(\mathcal{D}_3)|$ and a stability condition without strictly semistable objects. Then the maps are extended to the whole spaces by geometric arguments, so that the un-necessity of studying other hearts for computing the space $\mathrm{Stab}(\mathcal{D})$ comes as a consequence of the isomorphism.

Tameness then also means that any other bounded t-structure, which can be obtained by tilting a heart in $|\,\mathrm{EG}^\circ(\mathcal{D}_3)|$ at a torsion pair, can actually be obtained by the $\mathbb{C}$-action on $\mathrm{Stab}(\mathcal{D}_3)$, and that a connected component of the stability manifold is entirely covered by (and coincide with) the union of subsets of stability conditions supported on *reachable* hearts.

Last, the isomorphism 1.22 also implies that the sets of stability conditions supported on non-finite hearts have real co-dimension at least 1 in $\mathrm{Stab}(\mathcal{D}_3)$. This is the case for instance of the subset of stability conditions supported on the $\mathrm{Coh}\,\mathbb{P}^1$-shaped heart of the Ginzburg category associated with the Kronecker quiver, as we expect.

The period map. We focus again on the generic locus of spaces of differentials. Saddle connections that are dual to edges of triangulations correspond therefore to simple objects in the finite heart $\mathcal{H}$ corresponding to the triangulation. On the Riemann surface, we restrict to the space lying between the special trajectories connecting two zeroes and two poles. The choice of an orientation of the surface guarantees that the angle "measured by the differential" between a saddle connection γ connecting two zeroes and a generic horizontal trajectory connecting two poles is between 0 and π, and hence that $\int_\gamma \sqrt{\psi} \in \mathbb{H}$ for $\gamma \in \Gamma$. Identifying $\widehat{H}_1(\Psi) \simeq \Gamma \simeq K(\mathcal{H})$, the period map can be interpreted as a central charge $Z(\gamma) = \int_\gamma \sqrt{\Psi}$.

Counting semistable objects. The proof of isomorphism (1.4.5) by Bridgeland and Smith provides also a correspondence between saddle connections of a generic quadratic differential and stable objects of the corresponding stability condition. This, in turns, can be encoded in moduli spaces of stable representations of finite-dimensional algebras (in the abelian sense mentioned in 1.2.3), thanks to the work of King [38], and hence enumerated in appropriate sense. This opens new perspectives in classification and counting problems in the theory of flat surfaces. See [20, Theorem 1.4 and Section 1.6] for more details.

Note that we specify "generic" differential, i.e., we are not admitting counting strictly semistable objects. Note also that, at a triangulated level the notion of enumerative invariants, when defined, often requires Calabi-Yau dimension 3.

***A_2* example.** As an example, we work out explicitely the ingredients of the correspondence in the A_2 case. This is far from an exhaustive model given that all hearts of bounded t-structures are finite and appear in $\mathrm{EG}(\mathcal{D}_3(A_n))$. Let $\mathcal{D}_3(A_2)$ be the Ginzburg category of Calabi-Yau dimension 3 associated with the linear A_2 quiver

$$\bullet_1 \longrightarrow \bullet_2 \,.$$

The standard heart $\mathcal{H}_0 = \mathrm{rep}(A_2)$ has two simple objects denoted S_1, S_2, that are generators of the category, finitely many indecomposables, and five torsion classes. Let E be an indecomposable in $\mathcal{H}_0$ fitting into the sort exact sequence $S_2 \to E \to S_1$. The procedure of simple tilts gives rise to the following (partial) exchange graph (1.4.7), which is also the fundamental domain of $\mathrm{EG}(\mathcal{D}_3(A_2))$ with respect to the action of the spherical twist group $\mathrm{sph}(\mathcal{D}_3(A_2))$. Any $\mathcal{H}_i$, $i = 1, \dots, 4$, still has finitely many torsion pairs and two simple generators, so that two arrows should emanate from any $\mathcal{H}_i$ in the full exchange graph.

$$\begin{array}{ccccc} & & \mathcal{H}_1 = \langle S_1[1], E\rangle & \xrightarrow{\mu^\flat_E} & \mathcal{H}_2 = \langle S_2, E[1]\rangle \\ & \nearrow^{\mu^\flat_{S_1}} & & & \downarrow{\scriptstyle \mu^\flat_{S_2}} \\ \mathcal{H}_0 = \langle S_1, S_2\rangle & & & & \\ & \searrow_{\mu^\flat_{S_2}} & & & \\ & & \mathcal{H}_3 = \langle S_1, S_2[1]\rangle & \xrightarrow[\mu^\flat_{S_1}]{} & \mathcal{H}_4 = \langle S_1[1], S_2[1]\rangle \end{array} \tag{1.4.7}$$

The quiver A_2 can be obtained, with the procedure described in Definition 1.29, by a triangulation of the disc with one boundary component and five marked points on it. The iterior of the disc will contains three simply decorated points. See in Figure 1.1 examples of forward flips of triangulations (on the left), and un-decorated triangulations (on the right).

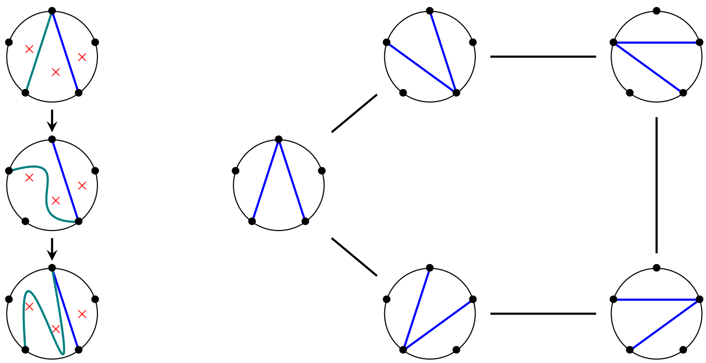

Figure 1.1. Disks with five marked points on the boundary (black dots), and three simply weighted decorations in the interior (red crosses). On the left: two forward flips, relative to the decorations, of the green edge of a triangulation. On the right: un-decorated triangulations and un-decorated flips.

A quadratic differentials that induces such a configuration with the procedure described in the appendix is a quadratic differential on the Riemann sphere $\mathbb{CP}^1$ with one pole of order 3 and three single zeroes. In a co-ordinate system centered in 0, it has therefore the form

$$\Psi(z) = (z - u_1)(z - u_2)(z - u_3)dz \otimes dz,$$

for three distinct points u_1, u_2, u_3 on $\mathbb{C}$. Here the pole is fixed at $\infty \in \mathbb{CP}^1$. As the triple (u_1, u_2, u_3) varies in $\mathbb{C}^3$ we get different quadratic differentials of the same form. The condition for the zeroes to be distinct can be reformulated as $\prod_{i<j}(u_i - u_j) \neq 0$. Of course the result is independent on permutations of u_1, u_2, u_3 so that the parameter space will be quotiented by the symmetric group Σ_3. Last we can assume that the centre of mass of these points is the origin, i.e., $u_1 + u_2 + u_3 = 0$. The meromorphic quadratic differential Ψ is equivalently specified by two parameters $a = (u_1u_2 + u_2u_3 + u_3u_1)$ and $b = -u_1u_2u_3$

$$\Psi_{a,b}(z) = (z^3 + az + b)dz \otimes dz,$$

for $4a^3 + 27b^2 \neq 0$. See [30] and [20, Section 12.1] for a precise description of the relevant space of quadratic differentials. Theorem 1.22 becomes the following statement (Theorem 1.24).

Theorem 1.24. *The connected component* $\mathrm{Stab}^\circ(\mathcal{D}_3(A_2))$ *is isomorphic to the universal cover* $\widetilde{M_3}$ *of the configuration space*

$$M_3 := \{(a,b) \in \mathbb{C}^2 \mid 4a^3 + 27b^2 \neq 0\}\,.$$

The isomorphism, specialized here to $n = 2$, $N = 3$, from the paper [30] by Ikeda, is first constructed from the space M_3 to the quotient $\mathrm{Stab}^\circ(\mathcal{D}_3(A_2))/\mathrm{sph}(\mathcal{D}_3(A_2))$, and then lifted using that $\pi_1(M_n) \simeq \mathrm{sph}(\mathcal{D}_3(A_2))$ both coincide with a braid group. In fact, there are several ways of computing $\mathrm{Stab}(\mathcal{D}_3(A_2))$ –see [**?**, 19] and [20, 30] for details, which also apply to other Calabi-Yau dimensions and other quivers, e.g., [45]. A fundamental domain of $\mathrm{Stab}^\circ(\mathcal{D}_3)$ with respect to the action of the spherical twist group consists in stability conditions supported on the finite hearts appearing in (1.4.7). The projection of the forgetful map from this fundamental domain to $\mathbb{R}^2$ coordinatized by the purely imaginary part of the central charge of S_1 and S_2 is given in Figure 1.2.

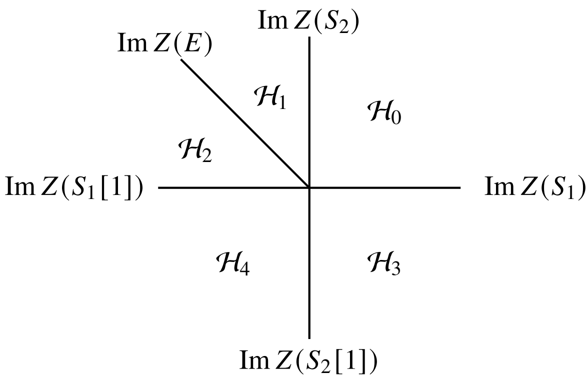

Figure 1.2. The projection of the forgetful map from $\mathrm{Stab}(\mathcal{D}_3(A_2))/\mathrm{sph}(\mathcal{D}_3(A_2))$ on the $\mathbb{R}^2$ plane with coordinates the purely imaginary part of the central charge of S_1 and S_2. It represents (the projection of) five chambers of $\mathrm{Stab}(\mathcal{D}_3(A_2))/\mathrm{sph}(\mathcal{D}_3(A_2))$ and of their walls.

A straight corollary of Theorem 1.24 is that the connected component $\mathrm{Stab}^\circ(\mathcal{D}_3(A_2))$, as a topological space, is contractible, [30, Theorem 7.13].

1.4.3 Generalizations

Few generalizations of the original Bridgeland-Smith correspondence (1.4.5) exist in literature. The most relevant ones concerne Ginzburg categories of Calabi-Yau dimension greater than 3 and Fukaya categories from flat surfaces.

The first is due to Ikeda [30], for the triangulated categories pvd $\Gamma_N(A_n)$ for $N \geq 3$. Similarly to the original Bridgeland-Smith result, it is based on a correspondence between hearts of bounded t-structures (up to the action of the N-spherical twist group) and N-angulations of a polygon with $(N-2)(n+1)+2$ edges. Simple tilts of hearts correspond to un-decorated flips of edges of the N-angulation and to cluster mutations in the coloured exchange graph. Ikeda's construction provides an additional example of the relation between central charges in the theory of stability condition and period maps, and shows that this is not limited to the CY_3 setup. Thanks to the relation between exchange graphs and coloured exchange graphs in cluster category theory, his approach seems to be adaptable to other Ginzburg algebras $\Gamma_N(Q, 0)$, $N \geq 3$, such that the corresponding N-cluster category admits a geometric description in terms of N-angulations. I am not aware of tentatives in this directions.

In a similar framework, in [4] the hypotheses of simple weights is relaxed and an un-punctured $\mathbf{S_w}$ is associated with an appropriate Verdier localization $\mathcal{D}(\mathbf{S_w})$ of a Ginzburg category. A union of connected components of $\mathrm{Stab}(\mathcal{D}(\mathbf{S_w}))$ is described in terms of quadratic differentials with vanishing order vector $\mathbf{w} \geq \mathbf{1}$.

The second is due to Haiden and collaborators, [27, 28]. They consider the more complicated setup of meromorphic quadratic differentials with exponential-type singularities and with only simple poles and zeroes. In the latter case a theory of counting finite-length geodesics is initiated from a Donaldson-Thomas theory enumerating semistable objects. The categories involved are Fukaya categories of surfaces with boundaries, whose objects correspond to a suitable collection of arcs. They are relevant in the context of Mirror Symmetry, and do not come from quivers with potential. Also in this context a period map plays the role of a central charge.

1.A Quivers, decorated marked surfaces, and quadratic differentials

1.A.1 Quivers with potential

Fix k an algebraicailly closed field, usually $k = \mathbb{C}$. We denote by (Q, W) a finite (possibly disconnected) oriented quiver (Q_0, Q_1, s, t) that has no loops or 2-cycles, with potential W, and up to right-equivalence. The completion of the path algebra kQ with respect to the bilateral ideal generated by arrows in Q_1 is denoted $\widehat{kQ}$. The lazy path (of length 0) at the vertex $j \in Q_0$ is denoted e_j. The potential W is a formal sum of cycles in $\widehat{kQ}$, up to cyclic equivalence, i.e., $\alpha_1\alpha_2\cdots\alpha_m$ with $t(\alpha_i) = s(\alpha_{i+1})$, for $i \in \mathbb{Z}/m\mathbb{Z}$, is equivalent to $\alpha_2\cdots\alpha_m\alpha_1$. The cyclic derivative with respect to an arrow $a \in Q_1$ is the unique k-linear map that takes a cycle of the form $c = uav$, $u, v \in \widehat{kQ}$, to $vu \in \widehat{kQ}$, and a cycle not containing a to 0. By ∂W we denote the ideal $\langle \partial_a W \mid a \in Q_1 \rangle \subset \widehat{kQ}$. In the examples we are most interested in, the potential involves all basic cycles, and the ideal ∂W is generated by monomials consisting on at least two letters. See [23] for these basic notions.

The *Jacobian algebra* $\mathcal{J}(Q, W)$ of a quiver with potential is the quotient of the complete path algebra $\widehat{kQ}$ with respect to the ideal ∂W. We assume it is a finite dimensional algebra. The category of finitely generated modules over $\mathcal{J}(Q, W)$ is denoted

$$\mathcal{A}(Q, W) := \mathrm{mod}\, \mathcal{J}(Q, W),$$

or $\mathcal{A}_Q$ for simplicity, and coincides with $\mathrm{rep}(Q, W)$, the category of finite dimensional representations of Q with relations induced by ∂W. It is a finite length, finite, abelian category, see for instance [34, Section 3]. Part of the information of the category $\mathcal{A}_Q$ is encoded in the combinatorics of the quiver with potential. The finite set of vertices $Q_0 = \{1, \ldots, n\}$ is in bijection with the set $\mathrm{Sim}(\mathcal{A}_Q) = \{S_j := e_j \mathcal{J} \mid j \in Q_0\}$ of simple objects of $\mathcal{A}_Q$. Their classes in the Grothendieck group form a basis of primitive vectors of

$$K(\mathcal{A}_Q) \simeq \mathbb{Z}^{|Q_0|}.$$

The dimension $\mathrm{ext}(S_i, S_j)$ of the extension group $\mathrm{Ext}_{\mathcal{A}_Q}(S_i, S_j)$ is given by the number of arrows q_{ji} from j to i.

A *mutation* μ_i of (Q, W) at a vertex i is an operation that creates a new quiver with potential $\mu_i(Q, W) = (\mu_i Q, \mu_i W)$ with the same set of vertices. The new set of arrows $(\mu_i Q)_1$ is constructed from Q_1 as follows:

(1) for any pair of arrows $a, b \in Q_1$, with $t(a) = i = s(b)$, add a new arrow $[ab] : s(a) \to t(b)$,
(2) replace any arrow with source or target i with the opposite arrow a^*,
(3) remove any 2-cycle.

The new potential $\mu_i W$ is define as $W' + W''$, where W' is obtained by replacing $[ab]$ any composition ab with $t(a) = i = s(b)$, and where $W'' = \sum_{a,b} [ab] b^* a^*$.

A quiver with potential (Q, W) is *non-degenerate* if any quiver with potential obtained from (Q, W) by iterated mutations has no loops or 2-cycles. Given a non-degenerate quiver with potential (Q, W), we fix an integer $N \geq 3$. We assume that either $N = 3$, or $N > 3$ and Q is acyclic.

The N-th *complete Ginzburg differentially graded (dg) algebra*

$$\Gamma_N := \Gamma_N(Q, W) := (\widehat{k\bar{Q}}, d)$$

is defined as follows, [26,33]. First introduce the graded quiver $\bar{Q}$ with vertices $\bar{Q}_0 = Q_0$ and graded arrows

- every $a : i \to j \in Q_1$, in degree 0,
- an opposite arrow $a^* : j \to i$ for any $a : i \to j \in Q_1$, in degree $-(N-2)$
- a loop e_i for any $i \in Q_0$, in degree $-(N-1)$.

Then the underlying graded algebra of Γ is the completion $\widehat{k\bar{Q}}$ of the graded path algebra $k\bar{Q}$ with respect to the ideal generated by the arrows of $\bar{Q}$. Finally, the differential d of Γ is the unique continuous linear endomorphism, homogeneous of degree 1, that satisfies the Leibniz rule and takes the following values

$$da = 0, \qquad da^* = \partial_a W, \qquad de_i = \sum_{a \in Q_1} e_i [a, a^*] e_i,$$

where e_i is the idempotent element at $i \in Q_0$ in kQ, i.e., $e_i^2 = e_i$, and $e_i \gamma$ (risp. γe_i) equals zero if $t(\gamma) \neq i$ (resp. $s(\gamma) \neq i$) and equals γ otherwise.

Remark 1.25. *The zero-th co-homology satisfies*

$$H^0\big(\Gamma_N(Q, W)\big) \simeq \mathcal{J}(Q, W).$$

Proof. Assume $N = 3$, then the arrows in $Q^{N=3}$ are in degree 0 (original arrows), $-(N-2) = -1$ (opposite arrows), and $-(N-1) = -2$ (loops). By definition of $H^0 = \frac{\ker d_0}{\operatorname{Im} d_1}$ we obtain that $H_0\big(\Gamma_N(Q, W)\big) = kQ/\partial W$. If there is no potential, we simply note that $H_0(\Gamma_N(Q)) = kQ$. ∎

Definition 1.26. *The* perfectly valued derived category *of the dg algebra* $\Gamma_N(Q, W)$ *is the full triangulated subcategory of the (unbounded) derived category* $\mathcal{D}(\Gamma_N(Q, W))$ *whose objects are dg modules of finite dimensional total cohomology. It is denoted* $\mathrm{pvd}(\Gamma_N(Q, W))$ *or* $\mathcal{D}_{fd}(\Gamma_N(Q, W))$.

Say $\Gamma_N := \Gamma_N(Q, W)$. The graded quiver $\bar{Q}$ is in facts the $\mathrm{Hom}^\bullet$-quiver of $\mathcal{D}(\Gamma_N)$ and $\mathrm{pvd}(\Gamma_N)$, viewed as triangulated categories generated by the dg modules $e_i\Gamma_N$: the graded arrows (with grading augmented by 1) $i \to j$ form a basis for $\mathrm{Hom}^\bullet(e_j\Gamma_N, e_i\Gamma_N)$. By remark 1.25 and [35], the derived category $\mathcal{D}(\Gamma_N)$ has a t-structure with heart $\mathrm{Mod}\,\mathcal{J}(Q, W)$ that restricts to $\mathrm{pvd}(\Gamma_N)$, on which it defines a bounded t-structure with heart $\mathrm{mod}\mathcal{J}(Q, W)$.

Two maior properties of $\mathrm{pvd}(\Gamma_N)$ are worth mentioning here. The first is that $\mathrm{pvd}(\Gamma_N)$ is Calabi-Yau of dimension N, i.e., for any objects E, F, there is a natural isomorphism of k-vector spaces $\mathrm{Hom}(E, F) \xrightarrow{\sim} \mathrm{Hom}(F, E[N])^\vee$. The second is that quivers with potential that are related by mutation at a vertex define equivalent perfectly derived categories, so that any quiver with potential which is mutation-equivalent to (Q, W) defines a bounded t-structure on $\mathrm{pvd}(\Gamma_N)$. In this perspective, simple tilts $\mu^\flat_{S_i}$ with respect to a simple object S_i are categorification of mutations with respect to the i-th vertex at the level of quivers. The picture is refined in a substantial way when the quiver with potential comes from a tiling of a decorated marked Riemann surface, Theorem 1.21.

1.A.2 Decorated marked surfaces and associated quivers

A way of constructing a quiver with potential is from a (undecorated) triangulation of a simply decorated marked surface. We briefly review here the definition of a weighted marked surface in the case with no internal marked points (*punctures*) nor decorations of weight $-1, 0$, and the notion of $\mathbf{w}$-mixed- and tri-angulations. From the latter we extract a quiver with potential. The following description is far from exhaustive and we refer to [40] for the original construction of quivers with potential from marked surfaces, to [50] for the refinement to decorated marked surfaces, and to [4] for the general definitions.

Definition 1.27. *A* decorated marked surface *(without punctures)* $(\mathbf{S}, \mathbb{M}, \Delta)$ *consists of*

- *a connected differentiable Riemann surface* $\mathbf{S}$*, with a fixed orientation and border* $\partial\mathbf{S} = \bigcup_{i=1}^{b} \partial_i$*,*
- *a non-empty finite set* $\mathbb{M}$ *of marked points on the boundary components, such that each connected component of* $\partial\mathbf{S}$ *contains at least one marked point, and*
- *a non-empty finite set* $\Delta = \{z_i\}_{i=1}^{r}$ *of points in the interior of* $\mathbf{S}$*.*

Up to homeomorphism, $(\mathbf{S}, \mathbb{M}, \Delta)$ is determined by the genus $g \geq 0$ of $\mathbf{S}$, the number b of boundary components and the integer partition $\mathbf{m} = (m_i)_{i=1}^{b}$ of the cardinality $|\mathbb{M}|$ of $|\mathbb{M}|$, where m_i is the number of marked points on ∂_i, and the integer r.

A *weight* function on Δ is a function $\mathbf{w}\colon \Delta \to \mathbb{Z}_{\geq -1}$. Here we assume it takes values in $\mathbb{Z}_{\geq 1}$. We say it is *compatible* with $\mathbf{S}$ and $\mathbb{M}$ if

$$\sum_{z\in\Delta} \mathbf{w}(z) - \sum_{i=1}^{b} (m_i + 2) \;=\; 4g - 4\,. \tag{1.A.1}$$

If $\mathbf{w}$, $\mathbb{M}$, and $\mathbf{S}$ are compatible, we will write $\mathbf{S_w}$ for the class of $(\mathbf{S}, \mathbb{M}, \Delta, \mathbf{w})$ up to diffeomorphism, and call this tuple a *weighted (decorated) marked surface*. For simplicity, we will not distinguish between $\mathbf{S_w}$ and a underlying Riemann surface $\mathbf{S}$.

We let $\mathbf{S}^\circ_{\mathbf{w}} := \mathbf{S_w} \setminus \partial\mathbf{S_w}$. An *open arc* is an isotopty class of curves $\gamma\colon [0,1] \to \mathbf{S_w}$ whose interior is in $\mathbf{S}^\circ_{\mathbf{w}} \setminus \Delta$ and whose endpoints are in the set of marked points $\mathbb{M}$, and an *open arc system* $\{\gamma_i\}$ is a collection of open arcs on $\mathbf{S_w}$ such that there is no (self-)intersection between any of them in $\mathbf{S}^\circ_{\mathbf{w}} \setminus \Delta$.

Definition 1.28. *A* $\mathbf{w}$-mixed-angulation *is a maximal open arc system and tiles* $\mathbf{S_w}$ *into polygons encircling a decoration of weight* w_i *exactly if they have* $w_i + 2$ *edges.*

The expression $\mathbf{w}$-*mixed*-angulation insists on the fact that polygons are allowed to have different shapes.

The *forward flip* at an arc γ of a $\mathbf{w}$-mixed-angulation is the operation that moves the endpoints of γ counter-clockwise along the adjacent sides of the smallest polygon encircling $\mathring{\gamma}$ and two decorations. The inverse operation is called a *backward flip*. These movements are *relative to the decorations*, so that, for instance, performing twice a forward flip at the same edge separating two triangles is not the identity. They transform a $\mathbf{w}$-mixed-angulation into another $\mathbf{w}$-mixed-angulation. See an example of a forward flip of a triangulation of a simply decorated disc with five marked points on the boundary in Figure 1.1.

The simply decorated case, i.e. $\mathbf{w} \equiv \mathbf{1}$, is studied e.g. in [37,51]. For this choice we write $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$ for $\mathbf{S}_{\mathbf{w}}$. A triangulation $\mathbb{T}$ of $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$ is a $(\mathbf{w} \equiv \mathbf{1})$-mixed-angulation, which in fact divides $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$ into triangles, each containing exactly one decoration.

The *exchange graph* $\mathrm{Exch}^{\circ}(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$ of a simply decorated marked surface is the directed graph whose vertices are triangulations of $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$ and whose oriented edges are forward flips between them, containing a vertex corresponding to a fixed initial triangulation. It is an infinite graph. The finite version $\mathrm{EG}(\mathbf{S}, \mathbb{M})$ is attached to the un-decorated version of $\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}}$. Vertices are tilings of $\mathbf{S}$ into triangles with endpoints in $\mathbb{M}$, up to isotopies in the interior of $\mathbf{S}$; arrows correspond to pairs of un-decorated flips (in the classical sense of Labardini-Fragoso), whose composition is an involution. In the un-decorated setup, tilings of $(\mathbf{S}, \mathbb{M})$ are called *dissections* for instance in [48], and N-angulations in the context of cluster and higher cluster categories.

Definition 1.29. *Given a triangulation $\mathbb{T}$ of a simply decorated marked surface, we associate to $\mathbb{T}$ a quiver with potential $(Q_{\mathbb{T}}, W_{\mathbb{T}})$ with the following procedure:*

- *the vertices of $Q_{\mathbb{T}}$ correspond to the open arcs in $\mathbb{T}$;*
- *the arrows of $Q_{\mathbb{T}}$ correspond to (anticlockwise) oriented intersection (at the endpoints) between open arcs in $\mathbb{T}$, so that there is a 3-cycle in $Q_{\mathbb{T}}$ locally in each triangle*
- *the potential $W_{\mathbb{T}}$ is the sum of all 3-cycles that locally coming from a triangle of $\mathbb{T}$ as above.*

Remark 1.30. *The quiver $Q_{\mathbb{T}}$ is finite and has no loops nor 2-cycles. Moreover any vertex has no more than* 2 *incoming and* 2 *out-going arrows, and its Jacobian albegra is finite dimensional. A forward or backward flip of the triangulation at a chosen arc induces a mutation of the associated quiver, though it is clear that the data of a quiver with potential and mutations contains less information than* $\mathrm{Exch}^{\circ}(\mathbf{S}_{\mathbf{w}\equiv\mathbf{1}})$.

1.A.3 Spaces of quadratic differentials on marked surfaces

Let $\mathbf{S}_g$ be a compact Riemann surface of genus g and $\omega_{\mathbf{S}_g}$ be its holomorphic cotangent bundle. A *meromorphic quadratic differential* Ψ on $\mathbf{S}_g$ is a meromorphic section of the line bundle $\omega^2_{\mathbf{S}_g}$. In local coordinates z on $\mathbf{S}_g$, it can be expressed as $\Psi(z) = f(z)dz \otimes dz$ for some meromorphic function f. A meromorphic quadratic differential Ψ on $\mathbf{S}_g$ has degree $4g-4$, which means that, if p_i denotes the zeroes of Ψ and q_j its poles, then $\sum \mathrm{ord}_\Psi(p_i) - \sum \mathrm{ord}_\Psi(q_j) = 4g-4$. The book by Strebel [55] is certainly the best reference for the theory of quadratic differentials. We refer nevertheless to [20, Section 3] and [36, Section 4] for (decorated) marked Riemann surfaces and triangulations associated to a meromorphic quadratic differential, and to [20] or to [4, Appendix A and B] and references therein for a quick introduction to the moduli spaces of quadratic differentials appearing in this article. We recall these notion briefly here.

- A saddle connection is a straigth arc connecting two zeroes along a fixed direction. Similarly to arcs before, it is defined up to isotopy. A quadratic differential is generic if it has no *horizontal* saddle connections.
- Near to a pole q_j of order at least 3 on $\mathbf{S}_g$, a quadratic differential Ψ defines exactly $\mathrm{ord}(q_j)-2$ distinguished trajectories in a fixed direction. Any generic Ψ on $\mathbf{S}_g$ with b poles of order $m_j+2 \geq 3$ and zeroes $p_1,\ldots,p_r$ induces a decorated marked surface $(\mathbf{S}, \mathbb{M}, \Delta)$ and a triangulation $\mathbf{T}$. The decorated marked surface is the real blow-up $\mathbf{S} = \mathrm{Bl}^{\mathbb{R}}_{q_1,\ldots,q_r} \mathbf{S}_g$ of $\mathbf{S}_g$ at the poles replacing a pole with a boundary component, with $\mathbb{M}$ being the set of distinguished trajectories and $\mathbf{m} = (m_j)$, $\Delta = \{p_1,\ldots,p_r\}$, and with a compatible weight function $\mathbf{w}$ defined by $w(p_i) = \mathrm{ord}_\Psi(p_i)$. The edges of the triangulation are horizontal trajectories of the quadratic differential connecting poles along distinguished directions.
- The moduli space of quadratic differentials (up to isomorphism) on a compact Riemann surface of genus g and with zeroes and pole order vector $(\mathbf{w}, \mathbf{m}+\mathbf{2})$ is denoted $\mathrm{Quad}_g(\mathbf{w},\mathbf{m})$.
- The standard double cover $(\widehat{\mathbf{S}_g}, \psi)$ of $(\mathbf{S}_g, \Psi)$ is the data of $\pi : \widehat{\mathbf{S}_g} \xrightarrow{2:1} \mathbf{S}_g$ such that $\pi^*\Psi = \psi^2$. If $\widehat{P}$ and $\widehat{Q}$ are the preimages of the sets of zeroes and poles of Ψ under π respectively, We let $\widehat{H}_1(\Psi)$ be the anti-invariant part of $H_1(\widehat{\mathbf{S}_g} \setminus \widehat{Q}, \widehat{Z}; \mathbb{C})$ with respect to the involution of $\widehat{\mathbf{S}_g}$ associated to π. Integrating ψ against a basis $\{\gamma_i\}_i$ of $\widehat{H}_1(\Psi)$ give local coordinate $\int_{\gamma_i} \psi$ on the moduli spaces of qadratic differentials $\mathrm{Quad}_g(\mathbf{w},\mathbf{m})$. This is the *period map*.
- Fix a finite rank free abelian group Γ and a reference surface $\mathbf{S}_\mathbf{w}$. A quadratic differential in $\mathrm{Quad}_g(\mathbf{w},\mathbf{m})$ is *period-framed* or Λ*-framed* if it is endowed with an isomorphism $\widehat{H}_1(\Psi) \simeq \Lambda$, so that we can define period coordinates valued in $\mathbb{C}^n = \mathrm{Hom}(\Gamma, \mathbb{C})$. It is said to be *Teichmüller-framed* if it equipped with a diffeomorphism $\mathbf{S}_\mathbf{w} \to \mathrm{Bl}^{\mathbb{R}}_{q_1,\ldots,q_r} \mathbf{S}_g$ preserving the marked points, the decorations, and their weights, up to diffeomorphism.

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